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Notes on *Ergodic Theory and Semisimple Groups*, by Robert J. Zimmer

Chapter 2

Semester Paper

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Abstract

This document provides a largely self-contained exposition of Chapter 2 from Zimmer's *Ergodic Theory and Semisimple Groups* [Zim84], which introduces ergodicity, basic notions of group actions on measurable spaces, and Moore's ergodicity theorem. The main contribution is a detailed reworking of the original proofs with explicit justifications for steps that were not fully worked out, along with clarification of assumptions that were implicit in the original text.

The appendix is divided into two parts: Part A contains the necessary background material and surrounding mathematical context needed for the main results, while Part B presents three observations that arose from my examination of some assumptions that were not completely clear to me in the original text. These observations, while not central to the main theory, provide additional insight into the technical details.

The goal is to offer an accessible and rigorous treatment that allows readers to engage with this beautiful material without requiring extensive background.

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Contents

Notation and conventions

I will use the following conventions.

$\mathbb{N}$	The set of natural numbers, not including 0.
$\overline{\mathbb{R}}$	The one-point compactification of $\mathbb{R}$, $\mathbb{R} \cup \{\infty\}$.

S^n	The sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ of dimension n.
T^n	The n-torus $\mathbb{R}^n/\mathbb{Z}^n$.

Let G be a group.

G_x	The stabilizer of $x \in X$ under the action of G on X.
Gx	The G-orbit of $x \in X$ under the left action of G on X.
$\mathrm{Z}(G)$	The center of G.

Let R be a ring.

$\mathrm{SL}(n, R)$	The group of invertible matrices of determinant 1 in R.
$\mathrm{SO}(n, R)$	The group of orthogonal matrices of determinant 1 in R.

Let S be a measure space with measure μ, and $1 \leq p < \infty$.

$\mathrm{L}^p(S)$	The space of μ-a.e. equivalence classes of measurable functions $f : S \to \mathbb{C}$ such that $\int_S \|f\|^p \, d\mu < \infty$.
$\mathrm{L}^\infty(S)$	The space of μ-a.e. equivalence classes of measurable functions $f : S \to \mathbb{C}$ that are bounded outside a set of measure 0.
χ_A	The characteristic function of the measurable set $A \subseteq S$.

Introduction

This text is concerned with actions of locally compact Hausdorff groups on measure spaces. In this setting, the central property we will study is *ergodicity*, whose definition will be our first focus.

In particular, we will first introduce some basic vocabulary and notions related to ergodicity, and explore some of the properties of group actions. This will constitute Chapter 1.

In Chapter 2, we restrict our attention to a particular class of actions (namely, lattices in suitable groups acting on suitable spaces), and the main theorem of the chapter (Moore's ergodicity theorem) will answer the question of when such actions are ergodic.

Throughout the rest of the introduction, we briefly present some of the examples of group actions that will be interesting to us.

Let G be a locally compact Hausdorff topological group. By Haar's theorem, G carries a left-invariant Radon measure (a Haar measure), unique up to scaling. A *lattice* in G is, roughly speaking, a discrete subgroup $\Gamma \leq G$ such that the quotient space G/Γ supports a finite G-invariant measure.

Two basic examples are:

- $\Gamma = \mathbb{Z}^n \leq \mathbb{R}^n = G$. The quotient $G/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus T^n, which is compact, hence carries a finite invariant measure.
- $\Gamma = \mathrm{SL}(n, \mathbb{Z}) \leq G = \mathrm{SL}(n, \mathbb{R})$. It is a classical theorem that $\mathrm{SL}(n, \mathbb{Z})$ is a lattice in $\mathrm{SL}(n, \mathbb{R})$ (we will not prove this here).

A second basic theme is that transitive group actions are the same thing as homogeneous spaces. Precisely, suppose G is σ-compact and acts continuously and transitively on a Hausdorff space X. Fix $x \in X$ and let G_x denote its stabilizer. The orbit map $G \to X$, $g \mapsto g \cdot x$, descends to a continuous bijection of G-spaces

$$G/G_x \longrightarrow X, \qquad gG_x \longmapsto g \cdot x,$$

which, under the standing hypotheses, is a homeomorphism. Thus every transitive G-space is (canonically up to conjugacy of the stabilizer) a homogeneous space G/H.

The following examples, with $G = \mathrm{SL}(n, \mathbb{R})$, are of particular interest.

- For $n = 2$, $\mathrm{SL}(2, \mathbb{R})$ acts on the complex upper half-plane by fractional linear transformations:

  $$\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

  This action is transitive, and identifies $\mathbb{H}^2 \simeq \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$, since the stabilizer of i is $\mathrm{SO}(2, \mathbb{R})$.

- More generally, for $n \geq 2$, $\mathrm{SL}(n, \mathbb{R})$ acts transitively on the space of positive-definite symmetric matrices:

  $$\mathcal{P}^1(n) = \big\{A \in M_n(\mathbb{R}) : A^{\mathrm{t}} = A,\ A > 0,\ \det A = 1\big\}, \qquad g \cdot A = gAg^{\mathrm{t}}.$$

  This realizes $\mathcal{P}^1(n)$ as the homogeneous space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$.

These actions induce actions on the boundaries of the spaces. For $\mathrm{SL}(2, \mathbb{R})$, the action extends continuously to the boundary $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ via the same fractional linear formulas (with the usual conventions at ∞). The stabilizer of ∞ is the subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} :\ a \in \mathbb{R}_{>0},\ b \in \mathbb{R} \right\} \subseteq \mathrm{SL}(2, \mathbb{R}),$$

and the orbit map identifies the boundary as the homogeneous space

$$\overline{\mathbb{R}} \cong \mathrm{SL}(2, \mathbb{R})/P.$$

For general n, $\mathrm{SL}(n, \mathbb{R})$ acts transitively on real projective space $\mathbb{R}\mathrm{P}^{n-1}$ by projective linear transformations. The stabilizer of the line $[\mathrm{e}_1] \in \mathbb{R}\mathrm{P}^{n-1}$ is the subgroup

$$\mathrm{SL}(n, \mathbb{R})_{[\mathrm{e}_1]} = \left\{ \begin{pmatrix} a & x \\ 0 & A \end{pmatrix} :\ a \neq 0, x \in \mathbb{R}^{n-1}, \det A = a^{-1} \right\} \subseteq \mathrm{SL}(n, \mathbb{R}),$$

and hence

$$\mathbb{R}\mathrm{P}^{n-1} \cong \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R})_{[\mathrm{e}_1]}.$$

Moore's ergodicity theorem will tell us that the action of any lattice $\Gamma \leq \mathrm{SL}(n, \mathbb{R})$ (for instance, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$) on $\mathbb{R}\mathrm{P}^{n-1}$ is ergodic.

Chapter 1

Ergodicity and smoothness

1.1 Introduction to ergodicity

Throughout this chapter, let G be a locally compact, Hausdorff, second countable topological group acting on a standard measurable space $(S, \mathscr{B})$ (see definition A.1.8 in the appendix) on the left. We assume that the action is measurable, meaning that the action map $G \times S \to S$, $(g, s) \mapsto gs$ is measurable. In this case, we write $G \curvearrowright S$ and call S a G-space.

Definition 1.1.1 Let μ be a σ-finite measure on S.

(a) μ is called quasi-invariant under the action of G (or G-quasi-invariant) if, for all $A \in \mathscr{B}$ and $g \in G$, $\mu(g^{-1}A) = 0$ if and only if $\mu(A) = 0$.

(b) μ is called invariant under the action of G (or G-invariant) if, for all $A \in \mathscr{B}$ and $g \in G$, $\mu(g^{-1}A) = \mu(A)$.

Remark 1.1.2 The action $G \curvearrowright S$ induces a G-action on the set of (σ-finite) measures on $(S, \mathscr{B})$, namely,

$$(g, \mu) \mapsto g_*\mu = \mu(g^{-1}\bullet).$$

An invariant measure on S is a fixed point of this action.

Recall that two measures are equivalent (see Definition A.2.3) if they have the same null sets. It is immediately verified that the G-action on the set of measures on S is well defined on measure classes, that is, G also acts on the set of (σ-finite) measure classes on S via

$$(g, [\mu]) \mapsto g_*[\mu] = [g_*\mu] = [\mu(g^{-1}\bullet)].$$

From this point of view, a measure μ is quasi-invariant if and only if its class $[\mu]$ is a fixed point under the G-action.

Finally, it is also important to note that any σ-finite measure is equivalent to a probability measure (see Remark A.2.4).

The main definition of the chapter is the following:

Definition 1.1.3 (Ergodicity) Let G and S be as above, and let μ be a quasi-invariant measure under G. The action of G on (S, μ) is called ergodic if every G-invariant measurable set is either null or conull. That is:

For any $A \in \mathscr{B}$, $gA = A$ for all $g \in G$ implies $\mu(A) = 0$ or $\mu(S \setminus A) = 0$.

Remark 1.1.4 An important fact to note is that if $G \curvearrowright (S, \mu)$ is ergodic, then $G \curvearrowright (S, \nu)$ will also be ergodic for any $\nu \sim \mu$. Therefore, in this sense, ergodicity is a question of measure classes.

(1.1.5) Essentially transitive and properly ergodic actions. We will prove later (Corollary 1.2.15) that orbits of the action $G \curvearrowright S$ are always measurable. With this in mind, we say that an action $G \curvearrowright (S, \mu)$ is essentially transitive if there exists a conull orbit, that is, an $x \in S$ such that $\mu(S \setminus Gx) = 0$.

An essentially transitive action is ergodic: if $A \in \mathscr{B}$ is G-invariant, then either $x \in A$, in which case $Gx \subseteq A$ and A is conull, or $x \notin A$, in which case A and Gx are disjoint, hence A is null.

An action is called properly ergodic if it is ergodic but not essentially transitive. In this case, ergodicity implies that every orbit is a null set.

Examples 1.1.6 (1) If $H \leq G$ is a closed subgroup, then there exists a unique G-invariant measure class on G/H (see theorem A.3.10 in the appendix). The action of G on G/H is transitive, hence ergodic.

(2) Suppose that S is a smooth manifold and that G acts on S by diffeomorphisms. If μ is a measure on S of the Lebesgue measure class (namely, locally in the same class as the measure induced by the Lebesgue measure), then μ is quasi-invariant, since diffeomorphisms preserve Lebesgue-null sets.

In particular, if G is a Lie group and H is a closed subgroup, the Lebesgue measure class on G/H is G-invariant, thus the unique G-invariant measure class of the previous example.

(3) (Irrational rotations of S^1) Let $\mathrm{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number, and consider the map $\mathrm{R}_\alpha : \mathrm{S}^1 \to \mathrm{S}^1$, $z \mapsto \mathrm{e}^{\mathrm{i}2\pi\alpha} z$. Then R_α generates an action $\mathbb{Z} \curvearrowright \mathrm{S}^1$ by

$$(k, z) \mapsto \mathrm{R}_\alpha^k(z) = \mathrm{e}^{\mathrm{i}2\pi k\alpha} z.$$

This action clearly preserves the arc length measure of S^1. It is not essentially transitive because all orbits are countable, hence null sets. However, it is ergodic (thus properly ergodic). Indeed: let $A \subseteq \mathrm{S}^1$ be invariant, and take $\chi_A(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ to be the $\mathrm{L}^2(\mathrm{S}^1)$-Fourier expansion of its characteristic function with respect to the Hilbert basis $(z^n)_{n \in \mathbb{Z}}$. Invariance implies:

$$\sum_{n \in \mathbb{Z}} a_n z^n = \chi_A(z) = \chi_A(\mathrm{e}^{\mathrm{i}2\pi\alpha} z) = \sum_{n \in \mathbb{Z}} a_n \mathrm{e}^{\mathrm{i}2\pi n\alpha} z^n,$$

so the Fourier coefficients coincide, $a_n = a_n e^{i2\pi n\alpha}$ for all $n \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Q}$, $a_n = 0$ for $n \neq 0$, which means that χ_A is constant almost everywhere, confirming ergodicity.

(4) Consider the usual action $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{R}^n$ by linear maps for $n \geq 2$. Since $\mathbb{Z}^n$ is invariant under this action, there is an induced action $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathrm{T}^n = \mathbb{R}^n/\mathbb{Z}^n$:

$$(\gamma, x + \mathbb{Z}^n) \mapsto \gamma x + \mathbb{Z}^n,$$

and this action is by automorphisms of T^n. The action $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{R}^n$ preserves the Lebesgue measure because determinants of elements of $\mathrm{SL}(n,\mathbb{Z})$ are all 1. Therefore, the induced action $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathrm{T}^n$ will preserve the Haar (Lebesgue) measure—more generally, though, any automorphism φ of a compact group K must be measure preserving (see Proposition A.3.5). We claim that it is also ergodic (hence, again, properly ergodic, because orbits are countable and countable sets in non-discrete groups are null, see Proposition A.3.4).

To show it, we see T^n as the product of n circles:

$$\begin{aligned}\mathrm{T}^n &= \{(x_1,\ldots,x_n) + \mathbb{Z}^n : x_1,\ldots,x_n \in \mathbb{R}\}\\ &\approx \{(e^{i2\pi x_1},\ldots,e^{i2\pi x_n}) : x_1,\ldots,x_n \in \mathbb{R}\} = \mathrm{S}^1 \times \cdots \times \mathrm{S}^1,\end{aligned}$$

and call $e^{i2\pi x} = (e^{i2\pi x_1},\ldots,e^{i2\pi x_n})$ for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ to abbreviate. In this form, the action of $\mathrm{SL}(n,\mathbb{Z})$ is given by

$$(\gamma, e^{i2\pi x}) \mapsto e^{i2\pi\gamma x}.$$

Now, for $A \subseteq \mathrm{T}^n$ an invariant measurable set under $\mathrm{SL}(n,\mathbb{Z})$, Fourier expand its characteristic function $\chi_A(e^{i2\pi x}) = \sum_{k\in\mathbb{Z}^n} a_k e^{i2\pi k^t x}$ in $\mathrm{L}^2(\mathrm{T}^n)$, where $k^t x$ denotes the scalar product. For $\gamma \in \mathrm{SL}(n,\mathbb{Z})$,

$$\chi_A(\gamma^{-1}\cdot e^{i2\pi x}) = \chi_A(e^{i2\pi\gamma^{-1}x}) = \sum_{k\in\mathbb{Z}^n} a_k e^{i2\pi k^t\gamma^{-1}x} = \sum_{k\in\mathbb{Z}^n} a_k e^{i2\pi((\gamma^t)^{-1}k)^t x},$$

so, changing the index of summation to $j = (\gamma^t)^{-1}k$, we finally obtain

$$\chi_A(\gamma^{-1}\cdot e^{i2\pi x}) = \sum_{j\in\mathbb{Z}^n} a_{\gamma^t j} e^{i2\pi j^t x}.$$

Invariance of A yields that for all $\gamma \in \mathrm{SL}(n,\mathbb{Z})$,

$$\sum_{k\in\mathbb{Z}^n} a_k e^{i2\pi k^t x} = \chi_A(e^{i2\pi x}) = \chi_A(\gamma^{-1}\cdot e^{i2\pi x}) = \sum_{k\in\mathbb{Z}^n} a_{\gamma^t k} e^{i2\pi k^t x},$$

hence $a_k = a_{\gamma^t k}$ for all $k \in \mathbb{Z}^n$ and $\gamma \in \mathrm{SL}(n,\mathbb{Z})$. However, for all $k \neq 0$, the set $\mathrm{SL}(n,\mathbb{Z})k = \{\gamma k : \gamma \in \mathrm{SL}(n,\mathbb{Z})\}$ is infinite (see Proposition A.9.1). Since $\sum_{\ell\in\mathrm{SL}(n,\mathbb{Z})k}|a_k|^2 = \sum_{\ell\in\mathrm{SL}(n,\mathbb{Z})k}|a_\ell|^2 \leq \sum_{k\in\mathbb{Z}^n}|a_k|^2 < \infty$, we obtain that $a_k = 0$. This implies that χ_A is constant almost everywhere, concluding the proof for ergodicity.

(5) Let $X = \{\pm 1\}^{\mathbb{N}}$. Then, X is a compact Abelian group, being the product of finite Abelian groups. The Haar measure on X is simply the product of the Haar measures on each factor. Let

$$H = \{x = (x_i)_{i\in\mathbb{N}} \in X : x_i = 1 \text{ for all but finitely many } i\}.$$

Then, H is a countable, dense subgroup of X. Moreover, the action of H on X by multiplication is ergodic (and, again, properly ergodic). To prove this fact, we resort to character theory (see appendix A.6). By Theorem A.6.2 and Proposition A.6.3, a Hilbert basis for $\mathrm{L}^2(X)$ is given by functions of the form $p_{i_1}\cdots p_{i_n}$, where $p_i : X \to \{\pm 1\} \subseteq \mathrm{S}^1$ is the projection on the i-th factor and $i_1, \dots, i_n$ is a (possibly empty) finite sequence of positive integers without repetitions.

It is clear that for every $i_1, \dots, i_n$ non-empty, there exists $h \in H$ such that $(p_{i_1}\cdots p_{i_n})(hx) = -(p_{i_1}\cdots p_{i_n})(x)$ for all $x \in X$. Hence, if $A \subseteq X$ is H-invariant and

$$\chi_A(x) = c + \sum c_{i_1,\dots,i_n}(p_{i_1}\cdots p_{i_n})(x),$$

then $\chi_A(hx) = \chi_A(x)$ a.e. for all $h \in H$. Uniqueness of Fourier coefficients implies $c_{i_1,\dots,i_n} = -c_{i_1,\dots,i_n} = 0$ for every non-empty $i_1, \dots, i_n$, so χ_A is constant almost everywhere, which proves ergodicity.

1.2 Smoothness

We first explore some properties of properly ergodic actions. The author rightfully points out that proper ergodicity is a phenomenon of complicated orbits. The first result is the following.

Proposition 1.2.1 *Suppose that S is a second countable topological space, that the action of G is continuous, and that μ is a quasi-invariant measure which is positive on open sets. If the action is properly ergodic, then, for almost every $s \in S$, Gs is a dense null set.*

Proof. Begin by observing that for every $W \subseteq S$ open, $\bigcup_{g\in G} gW$ is an open invariant set, so, by ergodicity, it must be conull. Therefore, if $\{W_i\}_i$ is a countable basis for the topology on S, the set

$$F = \bigcap_i \left(\bigcup_{g\in G} gW_i \right)$$

is conull, since its complement is the union of countably many null sets. Furthermore, every point $s \in F$ has a dense orbit, because said orbit intersects every W_i. Indeed: fix $s \in F$ and W_i. Then, $s \in \bigcup_{g\in G} gW_i$, so $s \in g_0 W_i$ for some g_0. This implies that $g_0^{-1}s \in W_i$. □

The following results will as well describe another sense in which proper ergodic actions induce complicated orbits. We first need a definition.

Definition 1.2.2 (Smooth action) Let S be a countably separated measurable G-space (see Definition A.1.5). The action of $G \curvearrowright S$ is called smooth if the quotient measurable structure on the orbit space S/G is countably separated.

We observe that smoothness is a regularity property on the orbit space. The following important proposition asserts that a proper ergodic action cannot be smooth.

Proposition 1.2.3 *Let $G \curvearrowright (S, \mu)$ be a smooth, ergodic action. Then there exists a conull orbit.*

Proof. First, take ν a probability measure on S equivalent to μ. It is also ergodic for the G-action, as observed in Remark 1.1.4. We will find a ν-conull orbit, which will also be μ-conull by equivalence.

Let $\mathscr{S} = \{A_i\}_i$ be a sequence of measurable sets separating points in S/G. We can assume that $\mathscr{S}$ is closed under taking complements —if not, just add to $\mathscr{S}$ all the complements of the A_i's. Let $p : S \to S/G$ be the quotient map, and $\tilde{\nu} = p_* \nu$. Note that for every measurable A in S/G, $p^{-1}(A) \subseteq S$ is a union of orbits, hence G-invariant. Therefore, $\tilde{\nu}(A) = \nu(p^{-1}(A)) \in \{0, 1\}$ by ergodicity.

Let $B = \bigcap\{A \in \mathscr{S} : \tilde{\nu}(A) = 1\}$, which is non-empty because $\mathscr{S}$ is closed under complements (hence there exists $A \in \mathscr{S}$ with full measure). B is a countable intersection of sets of measure 1, thus $\tilde{\nu}(B) = 1$. Now, it suffices to show that B consists of a single point, because, in that case, $p^{-1}(B)$ would be a conull orbit. If B contained two distinct points, x and y, there would exist some $A \in \mathscr{S}$ that separates them. Then, either A or its complement would have measure 1, implying that either x or y is not in B, a contradiction. □

Examples 1.2.4 As a quick consequence this proposition, we get that all the properly ergodic actions discussed in 1.1.6 are not smooth, namely:

(1) The action $\mathbb{Z} \curvearrowright \mathrm{S}^1$ by irrational rotations is not smooth.

(2) The action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathrm{T}^n$ is not smooth.

(3) The action $H \curvearrowright X = \{\pm 1\}^{\mathbb{N}}$ is not smoth.

The proof for Proposition 1.2.3 also shows the following. It is a small generalization of of the fact that invariant measurable functions under ergodic actions must be essentially constant.

Proposition 1.2.5 *Let $f : (S, \mu) \to (Y, \mathscr{C})$, where (S, μ) is an ergodic G-space and $(Y, \mathscr{C})$ is a countably separated space. If f is measurable and G-invariant (meaning $f(gs) = f(s)$ for all $g \in G$, $s \in S$), then f is constant almost everywhere, i.e., constant on a conull set.*

Proof. We only need to show that there exists a point $c \in Y$ such that $f^{-1}(c)$ is conull, or, equivalently,

$$f_*\mu(Y \setminus \{c\}) = \mu(f^{-1}(Y \setminus \{c\})) = \mu(S \setminus f^{-1}(c)) = 0,$$

that is, that $\{c\}$ is conull with respect to $f_*\mu$. The proof of 1.2.3 with $p = f$ shows this. □

In many situations, one deals with continuous actions. In those cases, the following proposition provieds a sufficient condition for smoothess.

Proposition 1.2.6 *Suppose that G acts continuously on a second countable Hausdorff topological space S. If every G-orbit is locally closed, then the action is smooth.*

Proof. Let $p : S \to S/G$ be the quotient map. We first claim that is open. Indeed: if $U \subseteq S$ is open, $p^{-1}(p(U)) = \bigcup_{g \in G} gU$ is also open, hence $p(U)$ is open in S/G.

Since S is second countable and p is open, S/G is second countable as well. Thus, to show that S/G with the Borel σ-algebra is countably separated, it suffices to see that any two points can be separated by open sets, because then any countable basis for the topology on S/G —a countable family of Borel sets— would separate points.

Take $x, y \in S$ and suppose that $p(x)$ and $p(y)$ are not separated by an open set. We will first show that $Gy \subseteq \overline{Gx}$. For this, choose $gy \in Gy$ and a neighborhood U of it. Then, $p(U)$ is open in S/G and contains $p(y)$ so it must also contain $p(x)$ by assumption. This means that there exists $z \in U$ such that $Gz = Gx$, so $z \in Gx \cap U$. Hence, $gy \in \overline{Gx}$. Similarly, $Gx \subseteq \overline{Gy}$.

In particular, Gy is dense in $\overline{Gx}$. Since Gx is locally closed, it is open in $\overline{Gx}$, so $Gy \cap Gx \neq \emptyset$, so $p(y) = p(x)$. □

This has the following immediate consequence.

Corollary 1.2.7 *If a compact group G acts continuously on a second countable Hausdorff space S, then the action is smooth.*

Proof. The hypotheses imply that orbits are compact. Compact sets in Hausdorff spaces are closed, hence locally closed. □

In the even more special case that S is a complete separable metrizable space, smoothness is equivalent to several regularity conditions on the orbits, one of them being that orbits are locally closed. The following theorem summarizes the situation.

Theorem 1.2.8 *Suppose that G acts continuously on a complete separable metrizable space S. Then, the following are equivalent:*

(i) The action is smooth.

(ii) All orbits are locally closed.

(iii) For each $s \in S$, the natural map $G/G_s \to Gs$ is a homeomorphism, where $Gs \subseteq S$ has the subspace topology.

The proof of this theorem relies on some lemmas, which we present and prove now.

This first lemma yields the equivalence between (ii) and (iii) by taking S to be the orbit closure.

Lemma 1.2.9 *With the hypotheses above, suppose $s \in S$ has a dense orbit. Then Gs is open if and only if the natural map $\varphi : G/G_s \to Gs$ is a homeomorphism.*

Proof. " $\Longleftarrow$ ": Suppose that $G/G_s \to Gs$ is a homeomorphism. Then, Gs is locally compact (because it is homeomorphic to a locally compact space) and Hausdorff (because it is a subspace of a Hausdorff space), and therefore satisfies the Baire category theorem. Moreover, since G is σ-compact, Gs is too, meaning that there exists a compact subset $A \subseteq Gs$ with non-empty interior. This is, there exists an open set $U \subseteq S$ such that $A \supseteq Gs \cap U$. But A is closed in S and Gs is dense, so we get the following:

$$Gs \supseteq A \supseteq \overline{Gs \cap U} \supseteq \overline{Gs} \cap U = U.$$

Thus, $Gs = GU$, which is open in S.

" $\Longrightarrow$ ": Suppose now that Gs is open in S. We know that $\varphi : G/G_s \to Gs$ is continous and bijective, so it only remains to see that it is open.

The first reduction that we can make is that it suffices to show openness for the action map $\tilde{\varphi} : G \to Gs$. Indeed: if $\tilde{\varphi}$ is open, take V an open set in G/G_s, this means that $q^{-1}(V)$ is open in G, where $q : G \to G/G_s$ is the quotient map. Now, $\varphi(V) = \tilde{\varphi}(q^{-1}(V))$, which is open in Gs.

The second reduction is that it suffices to check openness of $\tilde{\varphi}$ at the identity $e \in G$. In fact, we only need to show that if U is a neighborhood of e, then $\tilde{\varphi}(U)$ is a neighborhood of s in S. This is because left multiplication in G is a homeomorphism, for any open set $V \subseteq G$ and $g \in V$, $V = gU$ for U an open neighborhood of e. Then, $\tilde{\varphi}(V) = Vs = (gU)s = g(Us) = g\tilde{\varphi}(U)$, which is a neighborhood of gs in S whenever $\tilde{\varphi}(U)$ is a neighborhood of s, because G acts on S continuously. We conclude that $\tilde{\varphi}(V)$ is a neighborhood of all of its points, hence it is open.

The third and final reduction we can make is the following: it suffices to show that for any compact symmetric neighborhood U of $e \in G$, Us has non-empty interior. Namely, if this is the case, let N be any neighborhood of e. Take

U a compact symmetric neighborhood of e with $U^2 \subseteq N$. Then, if Us is a neighborhood of us for $u \in U$, then $u^{-1}Us$ will be a neighborhood of s, hence so will be Ns.

To show that Us has non-empty interior, choose a countable dense set $\{g_i\}_i \in G$. Then $Gs = \bigcup_i g_i Us$, a union of compact sets (which are closed since Gs is Hausdorff). The Baire category theorem holds for open subsets of complete metric spaces. In particular, it holds for Gs, so some $g_i Us$ has non-empty interior, hence so does Us. □

The statement (ii) $\implies$ (i) is Proposition 1.2.6. Hence, it only remains to establish the converse. So, suppose $s \in S$ with Gs dense in S, but such that Gs is not open. Following the same strategy as in the proof of the last Lemma, 1.2.9, we can further assume that Gs has empty interior. We aim to show that S/G is not countably separated.

Since any subset of a countably separated space is itself countably separated, it would suffice to find a non-countably separated space inside S/G. For this, we could just take an action—say, of a group H acting on a space X—which is already known to be non-smooth, together with an injective measurable map $\theta : X \to S$ that passes to the orbit spaces, and such that the induced map $X/H \to S/G$ is injective.

This strategy is promising because we have a natural candidate for such an X, namely the action described in Example 1.1.6 (5). Here, $X = \{\pm 1\}^{\mathbb{N}}$ is homeomorphic to the middle-thirds Cantor set. It is well known that the classical construction of the Cantor set in $[0, 1]$ generalizes easily, allowing for the construction of many injective continuous maps from X into any complete separable metric space.

However, instead of verifying full injectivity of the induced map $X/H \to S/G$, we will establish a slightly weaker condition that is nonetheless sufficient for our purposes, coming from the fact that H acts ergodically on X together with Proposition 1.2.5.

Lemma 1.2.10 *Let H be a group acting ergodically on (X, μ), where μ has no atoms (meaning that singletons have measure 0). If I is a measurable space and $f : X \to I$ is an H-invariant measurable map which is countable-to-one, then I is not countably separated.*

Proof. If I is countably separated, Proposition 1.2.5 implies that f is constant on a conull set. In particular, there exists a countable conull subset of X, contradicting the fact that μ has no atoms. □

Hence, we only need to find an injective continuous map $\theta : X \to S$ such that

(a) $\theta(Hx) \subseteq G\theta(x)$ for all $x \in X$, and

(b) $\theta(X)$ intersects each G-orbit in, at most, a countable set, because then, the map $X \to S/G$ would be H-invariant and countable-to-one, proving that S/G is not countably separated. Of course, the rest of the hypotheses of the lemma are satisfied: H here acts ergodically (see Example 1.1.6 (5)) and the measure on X, being the product of the normalized counting measures on each factor, has no atoms.

(1.2.11) Cantor space construction on S. For $x = (x_i) \in X$, write $p_n(x) = (x_1, \dots, x_n)$. Suppose that for each $x \in X$ and $n \in \mathbb{N}$, there is a non-empty open set $U(x,n) \subseteq S$ such that

(1) $\overline{U(x,n+1)} \subseteq U(x,n)$,

(2) $\operatorname{diam}(U(x,n)) \leq 1/n$,

(3) if $p_n(x) = p_n(y)$, then $U(x,n) = U(y,n)$, and

(4) if $p_n(x) \neq p_n(y)$, then $U(x,n) \cap U(y,n) = \emptyset$.

Then, for each x there exists a unique point in $\bigcap_{n\in\mathbb{N}} U(x,n)$, which we call $\theta(x)$. Indeed, existence is a consequence of the completeness of S: any sequence $(s_n)_n$ with $s_n \in U(x,n)$ is Cauchy by (2), therefore has a limit, which is in $\bigcap_{n\in\mathbb{N}} U(x,n)$ by (1). Uniqueness comes from the fact that any two points in $\bigcap_{n\in\mathbb{N}} U(x,n)$ must be arbitrarily close to each other by (2), hence equal. From (4), θ is injective. Finally, it is continuous: if the distance from y to x is small enough (namely, $p_n(y) = p_n(x)$ for some $n \in \mathbb{N}$), then from (2) and (3) it follows that the distance from $\theta(y)$ to $\theta(x)$ is smaller than $1/n$.

Of course, there are many possible choices for $U(x,n)$, but we aim to select them in such a way that conditions (a) and (b) are satisfied. Let $h_n = ((h_n)_i)_{i=1}^{\infty} \in X$ be given by

$$(h_n)_i = \begin{cases} 1, & \text{if } i \neq n \\ -1, & \text{if } i = n, \end{cases}$$

and suppose additionally that we can choose $U(x,n)$ so that

(5) For each $x \in X$, $n \in N$ there exists $g(x,n) \in G$ such that for all $k \leq n$,

$$U(xh_k, n) = g(x,k)U(x,n),$$

and

(6) There exists a neighborhood N of $e \in G$ such that for all $x, y \in X$ with $p_n(x) \neq p_n(y)$,

$$(N \cdot U(x,n)) \cap U(y,n) = \emptyset.$$

Then:

Lemma 1.2.12 *Condition (5) implies (a), and condition (6) implies (b).*

Proof. H is generated by the h_n's. Therefore, (5) $\Longrightarrow$ (a) follows from the fact that $\theta(xh_n) = g(x,n)\theta(x)$, which is obtained by passing condition (5) to the intersection.

Now, if condition (6) holds, it is clear that for every $x \in X$,

$$N\theta(x) \cap \theta(X) = \{\theta(x)\},$$

since for any $y \neq x$, there exists $n \in N$ such that $p_n(y) \neq p_n(x)$, and this implies that $(N \cdot U(x,n)) \cap U(y,n) = \emptyset$, but $N\theta(x) \subseteq N \cdot U(x,n)$, and $\theta(y) \in U(y,n)$, so $\theta(y) \notin N\theta(x)$.

Let M be a symmetric neighborhood of $e \in G$ such that $M^2 \subseteq N$, and let $\{g_i\}_i$ be a countable dense subset of G. We have that $\bigcup_i Mg_i = G$, since, for any $g \in G$, there exists $g_k \in Mg$, or, equivalently, $g \in Mg_k$. Because of this fact, $G\theta(x) = \bigcup_i Mg_i\theta(x)$, so we only need to see that $Mg_i\theta(x) \cap \theta(X)$ has at most one point for each i.

If $\theta(y) = m_1 g_i \theta(x)$ and $\theta(z) = m_2 g_i \theta(x)$ for $m_1, m_2 \in M$, we have that $\theta(y) = m_1 m_2^{-1}\theta(z) \in N\theta(z) \cap \theta(X)$, hence $\theta(y) = \theta(z)$. □

We now proceed to show that we can choose $U(x,n)$ such that (1)-(6) hold. We begin with the following little remark:

Lemma 1.2.13 *Let N be a compact symmetric neighborhood of $e \in G$. Then, for any $s \in S$ and any decreasing countable neighborhood basis $(W_i)_{i\in\mathbb{N}}$ of s,*

$$\overline{\bigcap_i NW_i} \subseteq Gs.$$

Proof. Let $t \in \overline{\bigcap_i NW_i}$, namely, $t = \lim_k t_k$ with $t_k \in \bigcap_i NW_i$. Write $t_k = g_k s_k$, with $g_k \in N$ and $s_k \in W_k$. Since the W_k's are a decreasing neighborhood basis of s, we have that $s_k \to s$. On the other hand, by passing to a subsequence, we can assume that $g_k \to g \in N$. Then, $t = gs$. □

We now proceed to construct the desired $U(x,n)$, $g(x,n)$ inductively. Fix a compact symmetric neighborhood N of $e \in G$. Let $\mathrm{I} \in X$ denote the identity, that is, $\mathrm{I}_i = 1$ for all i. Suppose that we have already chosen $U(x,k)$ and $g(x,k)$ for all $x \in X$ and $k \leq n$, so that (1)-(6) hold, and with the following additional assumptions:

(7) $s \in U(\mathrm{I},k)$,

(8) as a function of x, $g(x,k)$ depends only on $p_k(x)$,

(9) $g(xh_k,k) = g(x,k)^{-1}$, and

(10) for $k \leq n$, let $T_k : S \to S$ be defined by

$$T_k|_{U(x,k)} = g(x,k), \quad T_k|_{S\setminus\bigcup_{x,k} U(x,k)} = \text{identity}.$$

Then, the $g(x,k)$ are chosen so that H_n, the group of transformation of S generated by $\{T_k : k \leq n\}$, is finite Abelian, and acts simply transitively on the set $\{U(x,n) : x \in X\}$ (which is of cardinality 2^n).

For $n = 0$, we take $U(x, 0) = S$, $g(x, 0) = e$ for all x.

Now, let $(W_i)_i$ be a decreasing countable basis of neighborhoods of s (as in Lemma 1.2.13). Let $G_0 \subseteq G$ be the set of 2^n-fold products of elements of the form $g(x,k)$, $k \leq n$, which is finite. Observe that, as a consequence of (10), for every $T \in H_n$ and any $x \in X$, there exists some $g \in G_0$ such that $T|_{U(x,n)} = g$.

Then, by finiteness of G_0, $M = \bigcup_{g \in G_0} gNg^{-1}$ is a compact symmetric neighborhood of e, so, by Lemma 1.2.13 and the fact that Gs has empty interior, $\bigcap_i MW_i$ is nowhere dense. Thus, as a consequence of the Baire Category Theorem, there exists some fixed i such that MW_i is not dense in $U(\mathrm{I}, n)$. Since Gs is dense, we can pick $g(\mathrm{I}, n+1) \in G$ such that

$$g(\mathrm{I}, n+1)s \subseteq U(\mathrm{I}, n) \setminus \overline{MW_i}.$$

We can hence choose an open set $U(\mathrm{I}, n+1)$ satisfying the following:

- $s \in U(\mathrm{I}, n+1) \subseteq \overline{U(\mathrm{I}, n+1)} \subseteq U(\mathrm{I}, n)$.
- $U(\mathrm{I}, n+1) \subseteq W_i$
- $g(\mathrm{I}, n+1)\overline{U(\mathrm{I}, n+1)} \subseteq U(\mathrm{I}, n) \setminus MW_i$.
- $\mathrm{diam}(gU(\mathrm{I}, n+1)) \leq 1/(n+1)$ for all $g \in G_0 \cup G_0 g(\mathrm{I}, n+1)$ —the latter being a finite set.

The remaining choices are clear: given x, pick the unique $T \in H_n$ such that $U(x,n) = T(U(\mathrm{I}, n))$ and choose $g_0 \in G_0$ such that $T|_{U(\mathrm{I},n)} = g_0$. Define

$$U(x, n+1) = \begin{cases} g_0 U(\mathrm{I}, n+1), & \text{if } x_{n+1} = 1, \\ g_0 g(\mathrm{I}, n+1) U(\mathrm{I}, n+1), & \text{if } x_{n+1} = -1, \end{cases}$$

and

$$g(x, n+1) = \begin{cases} g_0 g(\mathrm{I}, n+1) g_0^{-1}, & \text{if } x_{n+1} = 1, \\ g_0 g(\mathrm{I}, n+1)^{-1} g_0^{-1}, & \text{if } x_{n+1} = -1. \end{cases}$$

It is now clear by construction that the conditions (1)-(10) are satisfied up to $n+1$. This finally gives us the existence of an injective continuous mapping $\theta : X \to S$ such that (a) and (b) are satisfied, hence a map f satisfying the conditions of Lemma 1.2.10.

We now sum up the proof of (i).

Proof. (Proof of Theorem 1.2.8) "(ii) $\iff$ (iii)": Given $s \in S$, take $S' = \overline{Gs}$ ($G \curvearrowright S'$ because the action of G on S is continuous). Applying Lemma 1.2.9 directly to the action $G \curvearrowright S'$ yields the result.

"(ii) $\implies$ (i)": This is Proposition 1.2.6.

"(i) $\implies$ (ii)": Suppose that Gs, for $s \in S$, is not locally closed. Then, again taking $S' = \overline{Gs}$, we have that Gs is not open in S'. By the previous Cantor space construction and Lemma 1.2.10, S'/G is not countably separated. Since subsets of countably separated spaces are countably separated (and the σ-algebra on S'/G is the subset σ-algebra with respect to S/G), we have that S/G is not countably separated, i.e., the action is not smooth. $\square$

The end of this section is devoted to the following remarkable theorem and some of its consequences.

Theorem 1.2.14 *Let S be a countably separated measurable G-space. Then, there is a compact metric space X on which G acts continuously and an injective measurable G-equivariant map $S \to X$.*

Corollary 1.2.15 *Let S be a countably separated measurable G-space. Then, orbits are measurable sets and stabilizers of points are closed subgroups.*

Proof. Since G is locally compact, Hausdorff, and second countable, it is σ-compact. In any continuous G-space, orbits are measurable sets when G is σ-compact.

By Theorem 1.2.14, there exists a compact metric space X with a continuous G-action and an injective measurable G-equivariant map $\varphi : S \to X$. Since X is a continuous G-space, all orbits in X are measurable.

For any $s \in S$, we have $\varphi(Gs) = G\varphi(s)$ by G-equivariance of φ. Since $G\varphi(s)$ is measurable in X and φ is measurable, the orbit $Gs = \varphi^{-1}(G\varphi(s))$ is measurable in S.

For stabilizers: since φ is injective and G-equivariant, $G_s = G_{\varphi(s)}$ for any $s \in S$. In the continuous action on X, stabilizers are closed subgroups, hence G_s is closed in G. $\square$

Corollary 1.2.16 *Any action of a compact group on a countably separated measurable space is smooth.*

Proof. By Theorem 1.2.14, there exists a compact metric space X with a continuous G-action and an injective measurable G-equivariant map $\varphi : S \to X$.

Since X is a compact metric space, it is second countable and Hausdorff. By Corollary 1.2.7, the action of the compact group G on X is smooth.

The G-equivariant map $\varphi : S \to X$ induces a well-defined map $\bar{\varphi} : S/G \to X/G$ on the orbit spaces. Since φ is injective, $\bar{\varphi}$ is also injective. Moreover, $\bar{\varphi}$ is measurable because φ is measurable and the quotient measurable structures are induced by the respective quotient maps.

Since X/G is countably separated (by smoothness of the action) and $\bar{\varphi}$ is an injective measurable map, S/G is also countably separated. Hence the action on S is smooth. □

This corollary shows that there is no "proper ergodic theory" for actions of compact groups.

Proof. (Proof of Theorem 1.2.14) Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of measurable sets in S separating points, and let χ_i denote the characteristic function of A_i.

Let B denote the unit ball in $\mathrm{L}^\infty(G)$, equipped with the weak-* topology, viewing $\mathrm{L}^\infty(G)$ as the dual of $\mathrm{L}^1(G)$ (this is possible because the Haar measure of G is σ-finite, since G is second countable). Then, by the Banach-Alaoglu theorem, B is compact. Additionally, $\mathrm{L}^1(G)$ is separable because G is. Therefore, since B is the closed unit ball of the dual of a separable Banach space, it is metrizable with the weak-* topology. Thus B is a compact metric space.

The group G acts on $\mathrm{L}^\infty(G)$ by left translations: $(g \cdot f)(h) = (\mathrm{L}_g f)(h) = f(g^{-1}h)$ for $f \in \mathrm{L}^\infty(G)$, $g, h \in G$. This action preserves the unit ball B and is continuous with respect to the weak-* topology. To see the continuity, note that left translation L_g on $\mathrm{L}^\infty(G)$ is the adjoint of right translation $\mathrm{R}_{g^{-1}}$ on $\mathrm{L}^1(G)$: for $f \in \mathrm{L}^\infty(G)$ and $\varphi \in \mathrm{L}^1(G)$,

$$\langle \mathrm{L}_g f, \varphi\rangle = \int_G f(g^{-1}h)\varphi(h)\, d\mu(h) = \int_G f(k)\varphi(gk)\, d\mu(k) = \langle f, \mathrm{R}_{g^{-1}}\varphi\rangle,$$

where the second equality uses the substitution $k = g^{-1}h$ and left-invariance of Haar measure. Since $\mathrm{R}_{g^{-1}}$ is continuous on $\mathrm{L}^1(G)$ (because it is an isometry), its adjoint L_g is weak-* continuous on $\mathrm{L}^\infty(G)$. Thus B is a compact metric G-space.

Let $X = \prod_{i=1}^\infty B$, equipped with the product topology. Since each B is compact and metric, X is compact and metrizable. The diagonal G-action on X given by $(g \cdot (f_i)_{i=1}^\infty) = (g \cdot f_i)_{i=1}^\infty$ is continuous, making X a compact metric G-space.

Define $\varphi : S \to X$ by $\varphi(s) = (\varphi_i(s))_{i=1}^\infty$, where each $\varphi_i(s) \in B$ is given by

$$[\varphi_i(s)](g) = \chi_i(g^{-1}s), \quad g \in G.$$

First, we verify that φ is G-equivariant. For any $g, h \in G$ and $s \in S$:

$$[\varphi_i(hs)](g) = \chi_i(g^{-1}(hs)) = \chi_i(g^{-1}hs),$$

while

$$[(\mathrm{L}_h\varphi_i(s))](g) = [\varphi_i(s)](h^{-1}g) = \chi_i((h^{-1}g)^{-1}s) = \chi_i(g^{-1}hs).$$

Thus $\varphi_i(hs) = \mathrm{L}_h\varphi_i(s)$ for all i, which means $\varphi(hs) = h \cdot \varphi(s)$.

Next, we show that φ is measurable. Since X has the product topology, it suffices to show that each coordinate map $\varphi_i : S \to B$ is measurable. For this, it suffices to show that for any $f \in \mathrm{L}^1(G)$, the map

$$s \mapsto \langle \varphi_i(s), f \rangle = \int_G f(g)[\varphi_i(s)](g)\, d\mu(g) = \int_G f(g)\chi_i(g^{-1}s)\, d\mu(g)$$

is measurable. Since $(s, g) \mapsto f(g)\chi_i(g^{-1}s)$ is measurable on $S \times G$, it follows from Fubini's theorem that $s \mapsto \int_G f(g)\chi_i(g^{-1}s)\, d\mu(g)$ is measurable.

Finally, we show that φ is injective. Suppose $s, t \in S$ and $\varphi(s) = \varphi(t)$. Then $\varphi_i(s) = \varphi_i(t)$ for all i, which means

$$[\varphi_i(s)](g) = [\varphi_i(t)](g) \text{ for almost all } g \in G$$

for each i. That is, $\chi_i(g^{-1}s) = \chi_i(g^{-1}t)$ for almost all $g \in G$, for each i.

Since there are only countably many sets A_i, the intersection

$$\bigcap_{i=1}^{\infty} \{g \in G : \chi_i(g^{-1}s) = \chi_i(g^{-1}t)\}$$

has full measure in G. Therefore, there exists some $g_0 \in G$ such that $\chi_i(g_0^{-1}s) = \chi_i(g_0^{-1}t)$ for all i. This means that $g_0^{-1}s \in A_i$ if and only if $g_0^{-1}t \in A_i$ for all i. Since the sequence $\{A_i\}_i$ separates points in S, we conclude that $g_0^{-1}s = g_0^{-1}t$, and therefore $s = t$. □

Chapter 2

Moore's ergodicity theorem

2.1 The question

We have seen some examples of ergodicity above. The central question of this text is whether or not certain naturally defined actions are ergodic, and this question will constitute the bulk of this chapter. For instance, we want to prove ergodicity of the boundary action of $\mathrm{SL}(2,\mathbb{Z})$ on $\overline{\mathbb{R}}$ given by fractional linear transformations, as described in the introduction, and more generally, actions of lattices in semisimple Lie groups. As remarked in the introduction, $\overline{\mathbb{R}}$ can be identified with $\mathrm{SL}(2,\mathbb{R})/P$, where P is the subgroup of upper triangular matrices. Hence, ergodicity of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\overline{\mathbb{R}}$ is a special case of the following question:

Question 2.1.1 *Let G be a semisimple Lie group, and $H_1, H_2 \leq G$ closed subgroups. When is H_1 ergodic on G/H_2?*

This itself is a special case of the following.

Question 2.1.2 *Let G be a semisimple Lie group, and S an ergodic G-space. If $H \leq G$ is a closed subgroup, when is H ergodic on S?*

When S is a topological space and the action $G \curvearrowright S$ is continuous and transitive, this is equivalent to Question 2.1.1. Indeed, since G is σ-compact, the orbit map $G/G_s \to S$ is a homeomorphism (see [Fol16]).

The following proposition is very useful.

Proposition 2.1.3 *Let G be a locally compact Hausdorff second countable group, S a G-space with quasi-invariant measure μ, and $H \leq G$ a closed subgroup. Then, H is ergodic on S if and only if G is ergodic on $S \times G/H$.*

Here, G acts on $S \times G/H$ diagonally: $g \cdot (s,x) = (gs, gx)$. The measure class on $S \times G/H$ is the product measure class.

Proof. "$\Longrightarrow$": Suppose H is ergodic on S. We prove that G is ergodic on $S \times G/H$ by contrapositive.

Assume G is not ergodic on $S \times G/H$. Then there exists a G-invariant measurable set $A \subseteq S \times G/H$ that is neither null nor conull.

For each $x \in G/H$, define the x-section $A_x = \{s \in S : (s, x) \in A\}$. We claim that G-invariance of A implies $gA_x = A_{gx}$ for all $g \in G$ and $x \in G/H$.

Indeed, $s \in gA_x$ if and only if $g^{-1}s \in A_x$, which holds if and only if $(g^{-1}s, x) \in A$. Since A is G-invariant, this is equivalent to $g \cdot (g^{-1}s, x) = (s, gx) \in A$, which means $s \in A_{gx}$. Thus $gA_x = A_{gx}$.

Since G acts transitively on G/H, we can write $G/H = \{g \cdot eH : g \in G\}$ where eH denotes the identity coset. By the relation above, for any $x = g \cdot eH$, we have $A_x = A_{g \cdot eH} = gA_{eH}$.

Now, if A_{eH} were null, then every section $A_x = gA_{eH}$ would be null (since the action preserves the measure class), and by Fubini's theorem, A would be null, contradicting our assumption. Similarly, if A_{eH} were conull, then every A_x would be conull, making A conull by Fubini's theorem.

Therefore, A_{eH} is neither null nor conull. But $A_{eH} = hA_{eH}$ for all $h \in H$, so A_{eH} is H-invariant. This contradicts the ergodicity of H on S.

"$\Longleftarrow$": Suppose G is ergodic on $S \times G/H$. We prove that H is ergodic on S again by contrapositive.

Assume H is not ergodic on S. Then there exists an H-invariant measurable set $B \subseteq S$ that is neither null nor conull.

By the existence of measurable sections (see A.1.12), we can choose a measurable section $\varphi : G/H \to G$ of the natural projection $p : G \to G/H$, so that $p(\varphi(x)) = x$ for all $x \in G/H$.

Define $A = \{(s, x) \in S \times G/H : s \in \varphi(x)B\}$. We claim that A is G-invariant.

Indeed, let $(s, x) \in A$, so $s \in \varphi(x)B$. For any $g \in G$, we have $\varphi(gx) = g\varphi(x)h$ for some $h \in H$. Since B is H-invariant, we have $\varphi(gx)B = g\varphi(x)hB = g\varphi(x)B$. Therefore, $gs \in \varphi(gx)B$ if and only if $gs \in g\varphi(x)B$, if and only if $s \in \varphi(x)B$, which shows $(gs, gx) \in A$ if and only if $(s, x) \in A$. Thus A is G-invariant.

Finally, since B is neither null nor conull, then by Fubini's theorem, A is also neither null nor conull, contradicting the ergodicity of G on $S \times G/H$. □

Corollary 2.1.4 *If $H_1, H_2 \leq G$ are closed subgroups of a locally compact Hausdorff second countable group, then H_1 is ergodic on G/H_2 if and only if H_2 is ergodic on G/H_1.*

Proof. By the above proposition, both statements are equivalent to G being ergodic on $G/H_1 \times G/H_2$. □

Moore's ergodicity theorem completely answers Question 2.1.1 for G a simple Lie group and H_1 or H_2 a lattice in G. It is formulated in a little more generality, so that it also provides a complete answer when G is a suitable semisimple Lie group and Γ is an irreducible lattice.

2.2 Irreducible lattices

Definition 2.2.1 (Irreducible lattice) Let G be a semisimple Lie group with finite center and $\Gamma \leq G$ a lattice. We say that Γ is irreducible if for every non-central normal closed subgroup (equivalently[1], every closed normal subgroup of positive dimension) N, Γ is dense when projected onto G/N.

This definition excludes examples such as $\Gamma_1 \times \Gamma_2 \leq G_1 \times G_2$, where the lattice decomposes as a product corresponding to a factorization of the ambient group. There are other characterizations of irreducibility for lattices, which show that a lattice is irreducible, roughly speaking, precisely when it does not come from such product constructions.

A typical example of an irreducible lattice is the following.

Examples 2.2.2 Let $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$. Then, $\mathrm{SL}(2,\mathcal{O})$ is an irreducible lattice in G, viewed as a subgroup of G via the map $g \mapsto (g, \sigma(g))$, where σ is the Galois map sending coefficients $a + b\sqrt{2}$ to $a - b\sqrt{2}$. For a discussion of this example, see [Zim84, §6.1].

We actually need a more general version of irreducibility, which we preface with a definition.

Definition 2.2.3 (Irreducible action) Let $G = G_1 \times \cdots \times G_n$ be a direct product, where G_i is a connected simple non-compact Lie group with finite center. Let S be an ergodic G-space with finite invariant measure. We say that the action of G on S is irreducible if for every non-central normal closed subgroup $N \leq G$, N is ergodic on S.

For instance, if G is simple, irreducibility is simply ergodicity.

Proposition 2.2.4 *Let G be as in Definition 2.2.3 and $\Gamma \leq G$ a lattice. Then, Γ is an irreducible lattice if and only if the action of G on G/Γ is irreducible.*

Proof. If $N \leq G$ is closed and normal, then N is ergodic on G/Γ if and only if Γ is ergodic on G/N (Corollary 2.1.4). This, in turn, is equivalent to Γ being dense when projected onto G/N by the following lemma. □

Lemma 2.2.5 *If $\Gamma \leq H$ is a subgroup of a locally compact Hausdorff (second countable) group H, then Γ is ergodic on H acting by left multiplication if and only if Γ is dense in H.*

[1] Indeed, every discrete normal subgroup is central; conversely, the center of G is discrete.

Proof. " $\Longrightarrow$ ": If $\overline{\Gamma} \neq H$, then $\overline{\Gamma}\backslash H$ is a non-trivial locally compact Hausdorff second countable group on which Γ acts trivially. Pick disjoint open sets U, V in $\overline{\Gamma}\backslash H$. Their preimages $\pi^{-1}(U), \pi^{-1}(V)$ under the canonical projection $\pi : H \to \overline{\Gamma}\backslash H$ are disjoint open sets in H, hence of positive Haar measure, thus neither null nor conull, and Γ-invariant, contradicting ergodicity.

" $\Longleftarrow$ ": See Remark B.0.4. □

2.3 Moore's theorem: statement and consequences

We are ready to state some versions of Moore's theorem and extract some consequences.

Theorem 2.3.1 (Moore's ergodicity theorem) *Let $G = G_1 \times \cdots \times G_n$ be a direct product of connected simple non-compact Lie groups with finite center, and $\Gamma \leq G$ an irreducible lattice. If $H \leq G$ is a closed subgroup and H is not compact, then H is ergodic on G/Γ.*

The proof of this theorem will constitute the final sections of this chapter. The converse assertion, namely, that if H is compact then H is not ergodic on G/Γ, is also true:

Corollary 2.3.2 *With H, Γ, G as in Theorem 2.3.1, Γ is ergodic on G/H if and only if H is not compact.*

Proof. Γ being ergodic on G/H is equivalent to H being ergodic on G/Γ by Corollary 2.1.4. Then, one direction is Theorem 2.3.1. For the other direction, suppose H is compact. Then, ergodicity of H on G/Γ implies that some H-orbit is conull by 1.2.3 and 1.2.7. This is impossible since the H-orbits are closed submanifolds of strictly smaller dimension, hence null. □

Examples 2.3.3 (1) If $\Gamma \leq \mathrm{SL}(2,\mathbb{R})$ is a lattice, then Γ is ergodic on $\overline{\mathbb{R}} \simeq \mathrm{SL}(2,\mathbb{R})/P$, since P is not compact.

(2) More generally, any lattice in $\mathrm{SL}(n,\mathbb{R})$ —like $\mathrm{SL}(n,\mathbb{Z})$— is ergodic on $\mathbb{RP}^{n-1}$.

(3) $\mathrm{SL}(n,\mathbb{Z})$ is ergodic on $\mathbb{R}^n$ by the natural action. Indeed, this is equivalent to $\mathrm{SL}(n,\mathbb{Z})$ being ergodic on $\mathbb{R}^n \setminus \{0\}$. The latter is a homogeneous space of $\mathrm{SL}(n,\mathbb{R})$, and for a non-zero vector $v \in \mathbb{R}^n$ its stabilizer is the subgroup

$$\mathrm{Stab}_{\mathrm{SL}(n,\mathbb{R})}(v) = \left\{ \begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix} : A \in \mathrm{SL}(n-1,\mathbb{R}) \right\},$$

which is non-compact: for $n \geq 3$ it already contains the non-compact group $\mathrm{SL}(n-1,\mathbb{R})$, while for $n = 2$ it contains the unipotent subgroup $\left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{R} \right\} \simeq (\mathbb{R}, +)$. Hence the hypotheses of Moore's theorem are met in every dimension $n \geq 2$, and the desired ergodicity follows.

(4) Let $\mathbb{H}^n$ be real hyperbolic n-space, realized as the unit ball endowed with the Poincaré metric, and put $G = \mathrm{Isom}^+(\mathbb{H}^n) \simeq \mathrm{SO}^+(n,1)$, which is a connected simple non-compact Lie group with finite center. Let $\Gamma \leq G$ be a lattice (so that, if Γ is torsion-free, $\mathbb{H}^n/\Gamma$ is a finite-volume hyperbolic manifold with fundamental group Γ). Then, every $g \in G$ extends to a conformal diffeomorphism of the boundary sphere S^{n-1} (for these facts about hyperbolic geometry, see [BP92]). Fixing a boundary point ξ, we obtain an identification $\mathrm{S}^{n-1} \simeq G/P$ with $P = \mathrm{Stab}_G(\xi)$. In the upper–half–space model one may take $\xi = \infty$; then P contains all horizontal translations $(x,t) \mapsto (x+b,t)$, $b \in \mathbb{R}^{n-1}$ (and vertical dilations), so P is non-compact. Moore's theorem applies, and Γ acts ergodically on S^{n-1}.

We will actually prove a more general version of Moore's theorem, which is stated in terms of irreducible actions instead of lattices.

Theorem 2.3.4 (Moore's ergodicity theorem) *Let $G = G_1 \times \cdots \times G_n$ be a direct product of connected simple non-compact Lie groups with finite center, and S an irreducible G-space with finite invariant measure. If $H \leq G$ is a closed subgroup and H is not compact, then H is ergodic on S.*

This implies Theorem 2.3.1 putting $S = G/\Gamma$.

2.4 Translation into a statement about unitary representations

Moore's theorem follows from a general fact about unitary representations of simple Lie groups. This section is devoted to describe this connection.

(2.4.1) Unitary representation associated to $G \curvearrowright S$. Let S be a G-space with finite invariant measure, where G is locally compact, Hausdorff, and second countable. For each $g \in G$, let $\pi(g) : \mathrm{L}^2(S) \to \mathrm{L}^2(S)$ be the unitary operator defined by $(\pi(g)f)(s) = f(g^{-1}s)$. Then, $\pi : G \to \mathcal{U}(\mathrm{L}^2(S))$ is a unitary representation (see Example A.5.5), called the *Koopman representation* of G on S.

(2.4.2) Ergodicity in terms of the representation. If $A \subseteq S$ is measurable and G-invariant, then $\chi_A \in \mathrm{L}^2(S)$ is G-invariant, hence so will be its projection f_A onto

$$\mathrm{L}^2_0(S) = \left\{ f \in \mathrm{L}^2(S) : \int_S f \mathrm{d}\mu = 0 \right\},$$

the orthogonal complement of $\mathbb{C}$ in $\mathrm{L}^2(S)$. If A is neither null nor conull, then χ_A is not constant, thus $f_A \neq 0$. Therefore, if G is not ergodic, there exist non-zero invariant vectors on $\mathrm{L}^2_0(S)$.

Conversely, suppose that G is ergodic on S and $f \in \mathrm{L}^2_0(S)$ is G-invariant. This means that for each $g \in G$, $f(s) = f(g^{-1}s)$ for a.e. $s \in S$ (such f is called *essentially invariant*). One would like to use Proposition 1.2.5 to conclude that f is constant a.e. on S, but notice that the proposition requires f to be strictly invariant, not essentially invariant. If G is countable, we could easily fix this by considering

$$S_0 = \bigcap_{g \in G} \{s \in S : f(g^{-1}s) = f(s)\}.$$

Then, S_0 is a G-invariant measurable set, and defining $\tilde{f}(s) = f(s)$ for $s \in S_0$ and $\tilde{f}(s) = 0$ for $s \notin S_0$, we have that $\tilde{f}$ is strictly G-invariant and $\tilde{f} = f$ a.e. We can apply Proposition 1.2.5 to conclude that $\tilde{f}$ is essentially constant, which implies that f is also essentially constant. Since $f \in \mathrm{L}^2_0(S)$, we must have $f = 0 \in \mathrm{L}^2_0(S)$. Thus, for G countable, ergodicity is equivalent to there being no non-zero invariant vectors in $\mathrm{L}^2_0(S)$.

The following lemma shows that the same is true for general G.

Lemma 2.4.3 *Let S be a G-space with quasi-invariant measure μ, and Y a countably generated measurable space. Suppose $f : S \to Y$ is measurable and essentially G-invariant (namely, that for all $g \in G$, $f(s) = f(g^{-1}s)$ for a.e. $s \in S$). Then, there exists a measurable function $\tilde{f} : S \to Y$ that is strictly G-invariant and $\tilde{f} = f$ a.e.*

Proof. Since Y is countably generated, it is measurably isomorphic to a Borel subset of $[0, 1]$ (see A.1.7); we henceforth regard $Y \subseteq [0, 1]$.

Let m be a left Haar measure on G. Define

$$S_0 = \{s \in S : \ g \mapsto f(g^{-1}s) \text{ is essentially constant on } G\}.$$

S_0 is measurable: Let λ be a probability measure on G equivalent to m. Define

$$I(s) = \int_G f(g^{-1}s) \, \mathrm{d}\lambda(g),$$

so $I : S \to [0, 1]$ is measurable by Fubini. Let

$$J(s) = \int_G |f(g^{-1}s) - I(s)| \, \mathrm{d}\lambda(g).$$

Again by Fubini, J is measurable, and $S_0 = J^{-1}(0)$. Hence S_0 is measurable.

Since f is essentially G-invariant, for each $g \in G$, $\mu\{s \in S : \ f(g^{-1}s) \neq f(s)\} = 0$. By Fubini's theorem, $m\{g \in G : \ f(g^{-1}s) \neq f(s)\} = 0$ for μ-a.e. s, so S_0 is conull.

Moreover, S_0 is G-invariant: if $s \in S_0$ and $h \in G$, then $g \mapsto f(g^{-1}(hs))$ is essentially constant on G. Indeed, we have $g \mapsto f(g^{-1}hs) = f((h^{-1}g)^{-1}s)$, which is essentially constant as g varies (being a composition of the essentially constant map $k \mapsto f(k^{-1}s)$ with the m-preserving bijection $g \mapsto h^{-1}g$). Thus $hs \in S_0$.

Finally, define

$$\tilde{f}(s) = \begin{cases} I(s), & s \in S_0, \\ y_0, & s \notin S_0, \end{cases}$$

where y_0 is any fixed element in Y. Then, $\tilde{f}$ is the desired G-invariant function that coincides with f μ-a.e. on S. □

Corollary 2.4.4 *If S is a G-space with finite invariant measure, then G is ergodic on S if and only if there are no non-zero G-invariant vectors in $\mathrm{L}^2_0(S)$.*

Remark 2.4.5 This result is no longer true if the measure on S is not finite, because for an invariant set A of infinite measure, χ_A will not be in $\mathrm{L}^2(S)$.

An example of this is $S = \mathbb{R}$ with Lebesgue measure and $G = \mathbb{Z}$ acting by translations $n \cdot x = x + n$. This action is clearly not ergodic. However, if $f \in \mathrm{L}^2_0(\mathbb{R})$ is $\mathbb{Z}$-invariant, we have $f(x+n) = f(x)$ for all $n \in \mathbb{Z}$, so f is 1-periodic. But a non-zero 1-periodic function cannot lie in $\mathrm{L}^2(\mathbb{R})$, hence the only $\mathbb{Z}$-invariant vector in $\mathrm{L}^2_0(\mathbb{R})$ is 0.

We also formulate the following result, generalizing Proposition 1.2.5.

Corollary 2.4.6 *If S is an ergodic G-space, Y is countably separated, and $f : S \to Y$ is measurable and essentially G-invariant, then f is essentially constant.*

By virtue of Corollary 2.4.4, Moore's theorem 2.3.4 follows from the following result:

Theorem 2.4.7 *Let $G = G_1 \times \cdots \times G_n$ be a direct product of connected simple non-compact Lie groups with finite center, and suppose π is a unitary representation of G (on a separable Hilbert space) so that for each G_i, $\pi|_{G_i}$ has no invariant vectors. If $H \leq G$ is a closed subgroup and $\pi|_H$ has a non-zero invariant vector, then H is compact.*

Indeed, assuming Theorem 2.4.7, take π to be the Koopman representation of G on $\mathrm{L}^2_0(S)$. Since the action of G on S is irreducible, each factor G_i is ergodic on S, hence by Corollary 2.4.4 there are no non-zero G_i-invariant vectors in $\mathrm{L}^2_0(S)$, so the hypothesis on $\pi|_{G_i}$ holds. Now if H is not compact, Theorem 2.4.7 implies that $\pi|_H$ has no non-zero invariant vectors; by Corollary 2.4.4 this is equivalent to ergodicity of H on S. This is precisely Theorem 2.3.4.

Theorem 2.4.7, in turn, is a consequence of the following result, known as the Howe-Moore vanishing of matrix coefficients theorem:

Theorem 2.4.8 (Howe-Moore) *Let G_i, G, $\pi : G \to \mathcal{U}(\mathcal{H})$ be as in Theorem 2.4.7. Then, all matrix coefficients of π vanish at ∞, namely, for any $\xi, \eta \in \mathcal{H}$,*

$$\langle \pi(g)\xi, \eta \rangle \to 0 \text{ as } g \to \infty.$$

Here $g \to \infty$ means that g leaves compact subsets of G.

Indeed, assuming Theorem 2.4.8, if $\xi \in \mathcal{H}$ is a non-zero invariant vector for $\pi|_H$, then the matrix coefficient $\langle \pi(g)\xi, \xi \rangle$ is constantly positive along H, hence H is compact, which proves Theorem 2.4.7.

The remaining two sections are dedicated to the proof of Howe-Moore's theorem, hence concluding our discussion.

2.5 Unitary representations of P

From now on, Hilbert spaces will be assumed to be separable.

It is necessary to develop first some necessary background in order to prove Theorem 2.4.8. We will first prove said theorem in the case $G = \mathrm{SL}(2, \mathbb{R})$, and then extend the result to the general case.

To study representations of $\mathrm{SL}(2, \mathbb{R})$, we first study representations of the upper triangular subgroup $P \leq \mathrm{SL}(2, \mathbb{R})$:

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0, b \in \mathbb{R} \right\}.$$

Then, we will use the fact that $\mathrm{SL}(2, \mathbb{R})$ is generated together by P and $\overline{P}$, the lower triangular subgroup. The representation theory of P that we need will follow from its structure as a semidirect product:

$$P = AN, \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \neq 0 \right\},$$

where $N \simeq (\mathbb{R}, +)$ is normal in P and $A \simeq (\mathbb{R}^*, \cdot) \leq P$.

The necessary background for this section is summarized in section A.7 of the appendix. We begin by studying briefly the representation theory of $\mathbb{R}^n$. Proofs of the following assertions can be found in [Dix69, Mac76, Loo53].

(2.5.1) Irreducible unitary representations of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Abelian (and because of Schur's Lemma), all its irreducible unitary representations are one-dimensional. Hence, the unitary representations of $\mathbb{R}^n$ are precisely its characters (see Section A.6). The characters of $\mathbb{R}^n$ are precisely the functions of the form

$$\lambda_\theta : \mathbb{R}^n \to \mathrm{S}^1, \quad t \mapsto \mathrm{e}^{\mathrm{i}\langle \theta, t \rangle},$$

for $\theta \in \mathbb{R}^n$. In other words, if $\widehat{\mathbb{R}^n}$ is the set of all characters of $\mathbb{R}^n$, then $\theta \mapsto \lambda_\theta$ is a group isomorphism $\mathbb{R}^n \simeq \widehat{\mathbb{R}^n}$.

(2.5.2) Direct integral models. Let μ be a σ-finite measure on $\widehat{\mathbb{R}^n}$ and let $(\mathcal{H}_\lambda)_{\lambda\in\widehat{\mathbb{R}^n}}$ be a (piecewise constant) field of Hilbert spaces over $\widehat{\mathbb{R}^n}$ (see Appendix A.7). Form the Hilbert space $\int^\oplus_{\widehat{\mathbb{R}^n}} \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)$ and define the representation $\pi_{(\mu,\mathcal{H}_\lambda)} : \mathbb{R}^n \to \mathcal{U}\left(\int^\oplus_{\widehat{\mathbb{R}^n}} \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)\right)$ by

$$(\pi_{(\mu,\mathcal{H}_\lambda)}(t)f)(\lambda) = \lambda(t)\, f(\lambda) \qquad (t \in \mathbb{R}^n,\ \lambda \in \widehat{\mathbb{R}^n}).$$

Then $\pi_{(\mu,\mathcal{H}_\lambda)}$ is a unitary representation acting fiberwise by multiplication by the character λ. Equivalently, this construction is the direct integral

$$\pi_{(\mu,\mathcal{H}_\lambda)} = \int^\oplus_{\widehat{\mathbb{R}^n}} \big(\dim \mathcal{H}_\lambda\big)\, \lambda \, \mathrm{d}\mu(\lambda),$$

where, for a representation σ and $n \in \mathbb{N} \cup \{0, \infty\}$, we use the shorthand

$$n\,\sigma := \bigoplus_{i=1}^{n} \sigma_i, \qquad \sigma_i = \sigma.$$

The following summarizes the representation theory of $\mathbb{R}^n$.

Proposition 2.5.3 *Let $\pi : \mathbb{R}^n \to \mathcal{U}(\mathcal{H})$ be a unitary representation.*

(1) There exist a σ-finite Borel measure μ on $\widehat{\mathbb{R}^n}$ and a field $(\mathcal{H}_\lambda)_{\lambda\in\widehat{\mathbb{R}^n}}$ of Hilbert spaces such that

$$\pi \;\simeq\; \pi_{(\mu,\mathcal{H}_\lambda)} \;=\; \int^\oplus_{\widehat{\mathbb{R}^n}} \big(\dim \mathcal{H}_\lambda\big)\, \lambda \, \mathrm{d}\mu(\lambda).$$

(2) If $\pi_{(\mu,\mathcal{H}_\lambda)}$ and $\pi_{(\mu',\mathcal{H}'_\lambda)}$ are two such direct–integral models, then they are unitarily equivalent if and only if

- *(a) $\mu \sim \mu'$ (the measures are equivalent), and*
- *(b) $\dim \mathcal{H}_\lambda = \dim \mathcal{H}'_\lambda$ for μ-a.e. λ (equivalently, for μ'-a.e. λ).*

We now consider groups having $\mathbb{R}^n$ as a normal subgroup. For that, we want to study the effect of an automorphism of $\mathbb{R}^n$ on its representation theory. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous group automorphism. Let $\alpha : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be its adjoint automorphism, given by

$$[\alpha(\lambda)](t) = \lambda(A^{-1}(t)).$$

Moreover, if $\pi : \mathbb{R}^n \to \mathcal{U}(\mathcal{H})$ is any unitary representation of $\mathbb{R}^n$, we let $\alpha(\pi) : \mathbb{R}^n \to \mathcal{U}(\mathcal{H})$ be the unitary representation

$$[\alpha(\pi)](t) = \pi(A^{-1}(t)).$$

If π is given in the form $\pi = \pi_{(\mu,\mathcal{H}_\lambda)}$, we wish to express $\alpha(\pi)$ in a similar form.

Proposition 2.5.4 *Let $A \in \mathrm{Aut}(\mathbb{R}^n)$, let $\alpha : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be its adjoint automorphism, and let $\pi = \pi_{(\mu,\mathcal{H}_\lambda)}$. Then:*

(1) $\alpha(\pi)$ is unitarily equivalent to $\pi_{(\alpha_*\mu,\,\mathcal{H}_{\alpha^{-1}\lambda})}$.

(2) If $V : \int_{\widehat{\mathbb{R}^n}}^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda) \to \int_{\widehat{\mathbb{R}^n}}^{\oplus} \mathcal{H}_{\alpha^{-1}\lambda} \,\mathrm{d}(\alpha_*\mu)(\lambda)$ is a unitary equivalence between $\alpha(\pi)$ and $\pi_{(\alpha_*\mu,\,\mathcal{H}_{\alpha^{-1}\lambda})}$, then, for every Borel set $E \subseteq \widehat{\mathbb{R}^n}$,

$$V\left(\int_E^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda)\right) \;=\; \int_{\alpha(E)}^{\oplus} \mathcal{H}_{\alpha^{-1}\lambda} \,\mathrm{d}(\alpha_*\mu)(\lambda).$$

Proof. (1) Define

$$T : \int_{\widehat{\mathbb{R}^n}}^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda) \longrightarrow \int_{\widehat{\mathbb{R}^n}}^{\oplus} \mathcal{H}_{\alpha^{-1}\lambda} \,\mathrm{d}(\alpha_*\mu)(\lambda), \qquad (Tf)(\lambda) = f\big(\alpha^{-1}\lambda\big).$$

Measurability of $\lambda \mapsto Tf(\lambda)$ is clear since α^{-1} is a Borel bijection of $\widehat{\mathbb{R}^n}$. Moreover,

$$\|Tf\|^2 = \int_{\widehat{\mathbb{R}^n}} \big\|f(\alpha^{-1}\lambda)\big\|^2 \,\mathrm{d}(\alpha_*\mu)(\lambda) = \int_{\widehat{\mathbb{R}^n}} \|f(\lambda)\|^2 \,\mathrm{d}\mu(\lambda),$$

so T is an isometry. Since α is bijective, $(T^{-1}g)(\lambda) = g(\alpha\lambda)$ defines the inverse map, hence T is unitary.

Put $\pi' = \pi_{(\alpha_*\mu,\,\mathcal{H}_{\alpha^{-1}\lambda})}$. For $t \in \mathbb{R}^n$ and $\lambda \in \widehat{\mathbb{R}^n}$,

$$\begin{aligned}
\big(T\,\alpha(\pi)(t)f\big)(\lambda) &= \big(\alpha(\pi)(t)f\big)(\alpha^{-1}\lambda) \\
&= (\alpha^{-1}\lambda)(A^{-1}t)\, f(\alpha^{-1}\lambda) \\
&= \lambda(t)\, f(\alpha^{-1}\lambda) \\
&= \big(\pi'(t)Tf\big)(\lambda),
\end{aligned}$$

using $(\alpha^{-1}\lambda)(A^{-1}t) = \lambda(t)$. Thus T intertwines $\alpha(\pi)$ with π', proving (1).

(2) Let V be a unitary equivalence as in the statement. Then $T^{-1}V$ is unitary and, for all $t \in \mathbb{R}^n$,

$$(T^{-1}V)\,\alpha(\pi)(t) \;=\; T^{-1}\,\pi'(t)\,V \;=\; \pi(t)\,(T^{-1}V),$$

i.e. $T^{-1}V$ commutes with every $\alpha(\pi)(t)$, equivalently with every $\pi(t)$.

Claim. $T^{-1}V$ commutes with every multiplication operator π_φ defined by

$$(\pi_\varphi f)(\lambda) = \varphi(\lambda)f(\lambda), \qquad \varphi \in L^\infty(\widehat{\mathbb{R}^n}).$$

Proof of the claim. Suppose $\varphi_N \to \varphi$ pointwise a.e., with $\|\varphi_N\|_\infty, \|\varphi\|_\infty \le M$. Then, for all $f, g \in \int^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda)$,

$$\begin{aligned}
\langle \pi_{\varphi_N} f, g\rangle &= \int_{\widehat{\mathbb{R}^n}} \varphi_N(\lambda)\, \langle f(\lambda), g(\lambda)\rangle \,\mathrm{d}\mu(\lambda) \\
&\xrightarrow[N\to\infty]{} \int_{\widehat{\mathbb{R}^n}} \varphi(\lambda)\, \langle f(\lambda), g(\lambda)\rangle \,\mathrm{d}\mu(\lambda) = \langle \pi_\varphi f, g\rangle,
\end{aligned}$$

by dominated convergence. Hence $\pi_{\varphi_N} \to \pi_\varphi$ in the weak operator topology. If π_{φ_N} commutes with a bounded operator A for all N, then passing to the limit yields $\pi_\varphi A = A\pi_\varphi$.

Therefore it suffices to prove the assertion for $\varphi \in C_c^\infty(\widehat{\mathbb{R}^n})$. Identify $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$. For N large enough, $\operatorname{supp}\varphi \subset (-N,N)^n$. Define φ_N to be the $(N\mathbb{Z})^n$-periodic function that agrees with φ on $[-N,N]^n$. Then $\varphi_N \to \varphi$ pointwise, so it suffices to prove the assertion for each φ_N.

Each φ_N is smooth on the torus $\mathbb{R}^n/(N\mathbb{Z})^n$, hence its Fourier series converges uniformly to φ_N. Consequently, the multiplication operators π_{φ_N} are operator–norm limits of finite linear combinations of the multipliers $\lambda \mapsto \lambda(t)$ (the trigonometric polynomials). Since $T^{-1}V$ commutes with every $\pi(t)$, it commutes with their finite linear combinations and with their operator–norm limits. Thus $T^{-1}V$ commutes with π_{φ_N} for all N, and by the first paragraph also with π_φ. This proves the claim.

Apply the claim with $\varphi = \chi_E$. Then π_{χ_E} is the orthogonal projection onto $\int_E^\oplus \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)$. Since $T^{-1}V$ commutes with π_{χ_E}, its range is invariant under $T^{-1}V$, i.e.

$$(T^{-1}V)\Big(\int_E^\oplus \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)\Big) \subseteq \int_E^\oplus \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda).$$

Applying T gives

$$V\Big(\int_E^\oplus \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)\Big) \subseteq T\Big(\int_E^\oplus \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)\Big) = \int_{\alpha(E)}^\oplus \mathcal{H}_{\alpha^{-1}\lambda} \, \mathrm{d}(\alpha_*\mu)(\lambda),$$

where the last equality follows from $(Tf)(\lambda) = f(\alpha^{-1}\lambda)$. Finally, applying the same argument to $V^{-1}T$ completes the proof. □

We now apply this proposition to representations of (locally compact, Hausdorff, second countable) groups having $\mathbb{R}^n$ as a normal subgroup. Suppose G is such a group. Then for each $g \in G$, conjugation by g gives an automorphism of $\mathbb{R}^n$. Therefore, we have an action of G on $\widehat{\mathbb{R}^n}$ given by $(g \cdot \lambda)(t) = \lambda(g^{-1}tg)$. Similarly, if $\pi : \mathbb{R}^n \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $\mathbb{R}^n$, we define $(g \cdot \pi)(t) = \pi(g^{-1}tg)$. Note that Proposition 2.5.4 applies separately for each g by letting A be conjugation by g.

Proposition 2.5.5 *Suppose $\mathbb{R}^n \leq G$ is a normal subgroup, and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation. Write $\pi|_{\mathbb{R}^n}$ as $\pi_{(\mu,\mathcal{H}_\lambda)}$ for some $(\mu, \mathcal{H}_\lambda)$ as in Proposition 2.5.3. Then:*

(1) μ is quasi-invariant under the action of G on $\widehat{\mathbb{R}^n}$.

(2) If $E \subseteq \widehat{\mathbb{R}^n}$ is Borel, let $\mathcal{H}_E = \int_E^\oplus \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)$. Then $\pi(g)\mathcal{H}_E = \mathcal{H}_{gE}$ for any $g \in G$.

(3) If π is irreducible, then μ is ergodic and $\dim \mathcal{H}_\lambda$ is constant μ-a.e.

Proof. (1) Fix $g \in G$. For $t \in \mathbb{R}^n$, $\pi(g^{-1}tg) = \pi(g)^{-1}\,\pi(t)\,\pi(g)$, so $\pi(g)$ implements a unitary equivalence between $\pi|_{\mathbb{R}^n}$ and $g \cdot (\pi|_{\mathbb{R}^n})$. By Proposition 2.5.4 (1) with $A = \mathrm{c}_g|_{\mathbb{R}^n}$, we have

$$g \cdot (\pi|_{\mathbb{R}^n}) \;\simeq\; \pi_{(g_*\mu,\, \mathcal{H}_{g^{-1}\lambda})}.$$

Since $\pi|_{\mathbb{R}^n} \simeq g \cdot (\pi|_{\mathbb{R}^n})$, it follows that $\pi_{(\mu,\mathcal{H}_\lambda)} \simeq \pi_{(g_*\mu,\, \mathcal{H}_{g^{-1}\lambda})}$. Applying Proposition 2.5.3 (2) to these two direct–integral models yields $\mu \sim g_*\mu$. Since $g \in G$ was arbitrary, μ is quasi-invariant under the G–action on $\widehat{\mathbb{R}^n}$.

In addition, Proposition 2.5.3 (2) also gives $\dim \mathcal{H}_\lambda = \dim \mathcal{H}_{g^{-1}\lambda}$ for μ-a.e. λ, i.e. the function $\lambda \mapsto \dim \mathcal{H}_\lambda$ is essentially G–invariant (and henceforth we assume $\lambda \mapsto \mathcal{H}_\lambda$ is essentially invariant under G. We will use this below).

(2) Fix $g \in G$. Since $\mu \sim g_*\mu$, let $\rho_g = \frac{\mathrm{d}\mu}{\mathrm{d}(g_*\mu)}$ be a Radon–Nikodym derivative on $\widehat{\mathbb{R}^n}$. Define

$$T_g : \int_{\widehat{\mathbb{R}^n}}^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda) \longrightarrow \int_{\widehat{\mathbb{R}^n}}^{\oplus} \mathcal{H}_{g^{-1}\lambda} \,\mathrm{d}(g_*\mu)(\lambda), \qquad (T_g f)(\lambda) = \rho_g(\lambda)^{1/2}\, f(\lambda).$$

Then T_g is an isometric surjection by construction and, for $t \in \mathbb{R}^n$,

$$(T_g\,\pi(t) f)(\lambda) = \rho_g(\lambda)^{1/2}\,\lambda(t) f(\lambda) = \lambda(t)\,(T_g f)(\lambda) = \big(\pi_{(g_*\mu,\, \mathcal{H}_{g^{-1}\lambda})}(t)\, T_g f\big)(\lambda),$$

so T_g intertwines $\pi_{(\mu,\mathcal{H}_\lambda)}$ with $\pi_{(g_*\mu,\, \mathcal{H}_{g^{-1}\lambda})}$. Consequently, $T_g\,\pi(g) :\; g \cdot \big(\pi|_{\mathbb{R}^n}\big) \simeq \pi_{(g_*\mu,\, \mathcal{H}_{g^{-1}\lambda})}$ is a unitary equivalence. Applying Proposition 2.5.4 (2) with $\alpha = g$ gives, for every Borel $E \subseteq \widehat{\mathbb{R}^n}$,

$$T_g\,\pi(g)\,\mathcal{H}_E \;=\; \int_{gE}^{\oplus} \mathcal{H}_{g^{-1}\lambda} \,\mathrm{d}(g_*\mu)(\lambda).$$

Finally, since $(T_g)^{-1}$ acts fiberwise by multiplication with $\rho_g^{-1/2}$, we have

$$(T_g)^{-1}\Big(\int_{gE}^{\oplus} \mathcal{H}_{g^{-1}\lambda} \,\mathrm{d}(g_*\mu)(\lambda)\Big) = \mathcal{H}_{gE}.$$

Therefore $\pi(g)\,\mathcal{H}_E = \mathcal{H}_{gE}$, as claimed.

(3) Suppose π is irreducible. If μ were not ergodic, there would exist a Borel G-invariant set $E \subseteq \widehat{\mathbb{R}^n}$ which is neither null nor conull. Then $\mathcal{H}_E = \int_E^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda)$ is a non-zero proper closed subspace of $\int^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu$, and, by part (2), it is G-invariant: $\pi(g)\mathcal{H}_E = \mathcal{H}_{gE} = \mathcal{H}_E$ for all $g \in G$. This contradicts the irreducibility of π. Hence μ is ergodic.

By the discussion after part (1), the function $\lambda \mapsto \dim \mathcal{H}_\lambda$ is essentially G-invariant. Since it is measurable, ergodicity of μ implies that it is essentially constant, i.e., constant μ-a.e. ∎

We are now ready to apply this discussion to the case of $P = AN$.

Theorem 2.5.6 *Let π be a unitary representation of $P = AN$ on $\mathcal{H}$. Then, one of the following holds:*

(1) $\pi|_N$ has non-trivial invariant vectors.

(2) For $g \in A$ and any vectors $\xi, \eta \in \mathcal{H}$, $\langle \pi(g)\xi, \eta\rangle \to 0$ as $g \to \infty$ (in A).

Corollary 2.5.7 *Let π be a unitary representation of P. Then any A-invariant vector is also P-invariant.*

Proof. (Proof of Corollary 2.5.7) Let $\mathcal{W} = \{\xi \in \mathcal{H} : \pi(n)\xi = \xi \text{ for all } n \in N\}$ be the subspace of N-invariant vectors. Since N is normal in P, for $p \in P$ and $n \in N$ we have $pnp^{-1} \in N$, hence $\pi(n)\,\pi(p)\xi = \pi(p)\,\pi(p^{-1}np)\xi = \pi(p)\xi$ for $\xi \in \mathcal{W}$. Thus $\mathcal{W}$ is P-invariant, and so is $\mathcal{W}^\perp$.

Consider the representation on $\mathcal{W}^\perp$. By definition it has no non-zero N-invariant vectors, so we are in case (2) of the theorem above. In particular, there are no A-invariant vectors in $\mathcal{W}^\perp$: if $\eta \in \mathcal{W}^\perp$ were A-invariant then $\langle \pi(a)\eta, \eta\rangle = \|\eta\|^2$ for all $a \in A$, contradicting the vanishing conclusion in (2).

Let $\xi \in \mathcal{H}$ be A-invariant and write $\xi = \xi_0 + \xi_1$ with $\xi_0 \in \mathcal{W}$ and $\xi_1 \in \mathcal{W}^\perp$. Since $\mathcal{W}$ and $\mathcal{W}^\perp$ are A-invariant, ξ_1 is also A-invariant; by the previous paragraph, $\xi_1 = 0$. Hence $\xi = \xi_0 \in \mathcal{W}$ is N-invariant. As A and N generate $P = AN$, ξ is P-invariant. □

Proof. (Proof of Theorem 2.5.6) Identify $N \simeq \mathbb{R}$. Write $\pi|_N = \pi_{(\mu,\mathcal{H}_\lambda)}$ as in Proposition 2.5.3.

If $\mu(\{0\}) > 0$, then $\mathcal{H}_{\{0\}} \neq \{0\}$ and consists of N-invariant vectors, so we are in case (1).

Assume now $\mu(\{0\}) = 0$. We prove (2). The action of A on N by conjugation is given by

$$g \cdot n_b \cdot g^{-1} = n_{a^2 b} \quad \text{for } g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \in A,$$

hence on $\widehat{N} \simeq \widehat{\mathbb{R}}$ we have (with our convention $g \cdot \lambda(t) = \lambda(g^{-1}tg)$)

$$g \cdot s = a^{-2}s, \qquad s \in \mathbb{R}.$$

Let $E, F \subset \mathbb{R} \setminus \{0\}$ be compact sets. Then for $g \in A$ with $|a|$ sufficiently large we have $(g \cdot E) \cap F = \emptyset$.

Fix unit vectors $f, h \in \int_{\mathbb{R}}^{\oplus} \mathcal{H}_\lambda \,\mathrm{d}\mu(\lambda)$ and $\varepsilon > 0$. Since $\mu(\{0\}) = 0$, there exist compact sets $E, F \subset \mathbb{R} \setminus \{0\}$ with

$$\|\chi_E f - f\| \leq \varepsilon, \qquad \|\chi_F h - h\| \leq \varepsilon.$$

(Indeed, since $\|f(\lambda)\|^2, \|h(\lambda)\|^2 \in L^1(\mu)$ and $\mu(\{0\}) = 0$, choose $0 < \delta < M$ so that $\int_{|\lambda|<\delta} \|f(\lambda)\|^2 \, \mathrm{d}\mu + \int_{|\lambda|>M} \|f(\lambda)\|^2 \, \mathrm{d}\mu < \varepsilon^2$, and set $E = \{\delta \leq |\lambda| \leq M\}$; similarly for h.) Then, for any $g \in A$,

$$\begin{aligned}\big|\langle \pi(g)f, h\rangle - \langle \pi(g)(\chi_E f), \chi_F h\rangle\big| &= \big|\langle \pi(g)(f - \chi_E f), h\rangle + \langle \pi(g)\chi_E f, h - \chi_F h\rangle\big| \\ &\leq \|\pi(g)(f - \chi_E f)\| \, \|h\| + \|\pi(g)\chi_E f\| \, \|h - \chi_F h\| \\ &\leq \|f - \chi_E f\| + \|h - \chi_F h\| \leq 2\varepsilon.\end{aligned}$$

By Proposition 2.5.5 (2), $\pi(g)\,\mathcal{H}_E = \mathcal{H}_{gE}$. Choosing $g \in A$ with $gE \cap F = \emptyset$, we have $\mathcal{H}_{gE} \perp \mathcal{H}_F$, hence

$$\langle \pi(g)(\chi_E f), \chi_F h\rangle = 0.$$

Therefore $|\langle \pi(g)f, h\rangle| \leq 2\varepsilon$ for all such g. Since $\varepsilon > 0$ was arbitrary, $\langle \pi(g)f, h\rangle \to 0$ as $g \to \infty$ in A. This is (2). □

2.6 Vanishing of matrix coefficients

We begin with the proof of Howe-Moore's Theorem (Theorem 2.4.8) for $G = \mathrm{SL}(2, \mathbb{R})$, which will be used to prove the general case.

(2.6.1) Cartan decomposition. Let us first recall the polar decomposition of a matrix. If $T \in \mathrm{SL}(n, \mathbb{R})$, then we can write $T = US$ for U orthogonal and S symmetric positive definite. Since S is symmetric, it is orthogonally diagonalizable, namely, there exists an orthogonal matrix U_0 such that $S = U_0 D U_0^{-1}$, where D is diagonal and its diagonal entries are positive. Hence, any $T \in \mathrm{SL}(n, \mathbb{R})$ has an expression $T = U_1 D U_2$, where $U_i \in \mathrm{SO}(n, \mathbb{R})$ and D is a positive diagonal matrix. Thus, we can write

$$\mathrm{SL}(n, \mathbb{R}) = KAK,$$

where $K = \mathrm{SO}(n, \mathbb{R})$ is compact and A is the group of positive diagonals. This is called the Cartan decomposition of $\mathrm{SL}(n, \mathbb{R})$.

Lemma 2.6.2 *Let G be a (locally compact Hausdorff second countable) group admitting a decomposition $G = KAK$ with K compact and A a closed subgroup. Let π be a unitary representation of G. If for all matrix coefficients f of π one has $f(a) \to 0$ as $a \to \infty$, $a \in A$, then all matrix coefficients of π vanish at ∞ on G.*

Proof. Fix $\xi, \eta \in \mathcal{H}$ and set $f(g) = \langle \pi(g)\xi, \eta\rangle$. Note that for $g = k_1 a k_2$ we have

$$f(g) = \langle \pi(g)\xi, \eta\rangle = \langle \pi(a)\,\pi(k_2)\xi,\, \pi(k_1)^{-1}\eta\rangle.$$

Suppose by contradiction that f does not vanish at ∞ on G. Then there exist $\varepsilon > 0$ and $g_n \to \infty$ in G with $|f(g_n)| \geq \varepsilon$ for all n. Write $g_n = k_{1,n} a_n k_{2,n}$ with

$k_{i,n} \in K$, $a_n \in A$. By compactness of K, passing to a subsequence, we may assume $k_{2,n} \to k$ and $k_{1,n}^{-1} \to k'$ in K.

Since π is strongly continuous, we have $\pi(k_{2,n})\xi \to \pi(k)\xi$ and $\pi(k_{1,n})^{-1}\eta \to \pi(k')\eta$. Hence for n large,

$$\left|\langle \pi(g_n)\xi, \eta\rangle - \langle \pi(a_n)\pi(k)\xi, \pi(k')\eta\rangle\right| < \tfrac{\varepsilon}{2},$$

so $\left|\langle \pi(a_n)\xi', \eta'\rangle\right| \geq \varepsilon/2$ with $\xi' = \pi(k)\xi$, $\eta' = \pi(k')\eta$.

Finally, if (a_n) were contained in a compact subset of A, then (g_n) would be contained in the compact set $K \cdot (\overline{\{a_n\}}) \cdot K$, contradicting $g_n \to \infty$. Thus $a_n \to \infty$ in A, and we have found a matrix coefficient along A that does not vanish at ∞. This contradicts the hypothesis, completing the proof. □

We prove now Theorem 2.4.8 for $G = \mathrm{SL}(2, \mathbb{R})$.

Theorem 2.6.3 *If π is a unitary representation of $G = \mathrm{SL}(2, \mathbb{R})$ with no invariant vectors, then all matrix coefficients of π vanish at ∞.*

Proof. By Lemma 2.6.2, it suffices to prove vanishing along the diagonal subgroup A. By Theorem 2.5.6, it is enough to show that $\pi|_N$ has no non-zero invariant vector.

Suppose, towards a contradiction, that $0 \neq \xi \in \mathcal{H}$ is N-invariant. Define $f : G \to \mathbb{C}$ by $f(g) = \langle \pi(g)\xi, \xi\rangle$. Then f is continuous and bi-N-invariant: for $n_1, n_2 \in N$,

$$f(n_1 g n_2) = \langle \pi(n_1)\pi(g)\pi(n_2)\xi, \xi\rangle = \langle \pi(g)\xi, \xi\rangle = f(g).$$

Right N-invariance implies that f descends to a continuous function $\varphi : G/N \to \mathbb{C}$, and left N-invariance makes φ N-invariant for the left action of N on G/N.

In $G = \mathrm{SL}(2, \mathbb{R})$, N is the stabilizer of $e_1 = (1, 0)^{\mathrm{t}}$ under the natural linear action on $\mathbb{R}^2$. Hence the orbit map $G \to \mathbb{R}^2 \setminus \{0\}$, $g \mapsto g \cdot e_1$, induces a G-equivariant homeomorphism $G/N \simeq \mathbb{R}^2 \setminus \{0\}$. Under this identification, the left action of N on G/N corresponds to the usual matrix multiplication of N on $\mathbb{R}^2 \setminus \{0\}$.

The N-orbits in $\mathbb{R}^2 \setminus \{0\}$ are precisely: (i) each horizontal line $\{(x, y) : y = c\}$ with $c \neq 0$, and (ii) the points on the x-axis $\{(x, 0) : x \neq 0\}$ (since N acts by $(x, y) \mapsto (x + by, y)$). Any continuous function on $\mathbb{R}^2 \setminus \{0\}$ which is constant on these orbits must be constant on the x-axis.

Under $G/N \simeq \mathbb{R}^2 \setminus \{0\}$, the x-axis corresponds to $P/N \subseteq G/N$. Hence φ is constant on P/N, i.e., f is constant on P. Since π is unitary, for any $p \in P$ we have $\|\pi(p)\xi\| = \|\xi\|$, and by Cauchy–Schwarz,

$$|\langle \pi(p)\xi, \xi\rangle| \leq \|\pi(p)\xi\|\,\|\xi\| = \|\xi\|^2.$$

Constancy of f on P gives $\langle \pi(p)\xi, \xi \rangle = \|\xi\|^2$, so we have Cauchy–Schwarz equality, hence $\pi(p)\xi = c(p)\,\xi$ with $|c(p)| = 1$. Plugging back into f yields $c(p) = 1$, hence $\pi(p)\xi = \xi$ for all $p \in P$. Thus ξ is P-invariant, and consequently f is bi-P-invariant.

Finally, P has a dense orbit on G/P (identify $G/P \simeq \mathbb{RP}^1$ and note that P acts transitively on the open cell), hence a continuous bi-P-invariant function on G must be constant. Therefore f is constant on G, which forces ξ to be G-invariant, contradicting the hypothesis. This proves that $\pi|_N$ has no non-zero invariant vector, and the theorem follows. □

Now, we prove it for $G = \mathrm{SL}(n, \mathbb{R})$.

Theorem 2.6.4 *If π is a unitary representation of $G = \mathrm{SL}(n, \mathbb{R})$ with no invariant vectors, then all matrix coefficients of π vanish at ∞.*

Proof. Let $A \leq G$ be the subgroup of diagonal matrices. We write an element $a \in A$ as $a = (a_1, \ldots, a_n)$, meaning $a = \mathrm{diag}(a_1, \ldots, a_n)$, $\prod_{i=1}^n a_i = 1$. Let B be the set of matrices $b = (c_{ij})$ with $c_{ii} = 1$, $c_{ij} = 0$ for $i \geq 2$, $i \neq j$, namely,

$$B = \left\{ b = \begin{pmatrix} 1 & b_2 & b_3 & \cdots & b_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : b_i \in \mathbb{R} \right\}.$$

We denote such an element by $b = (1\ b_2\ \cdots\ b_n)$. A direct calculation shows that for $a = \mathrm{diag}(a_1, \ldots, a_n) \in A$ and $b \in B$,

$$a\,b\,a^{-1} = \begin{pmatrix} 1 & \frac{a_1}{a_2} b_2 & \frac{a_1}{a_3} b_3 & \cdots & \frac{a_1}{a_n} b_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in B.$$

Hence $aBa^{-1} = B$ for all $a \in A$. It follows that $H = AB$ is a subgroup of G and that B is normal in H. The group B is isomorphic to $\mathbb{R}^{n-1}$ via $b \leftrightarrow (b_2, \ldots, b_n)$. By Lemma 2.6.2, it suffices to prove that the matrix coefficients of $\pi|_A$ vanish at ∞. In the $\mathrm{SL}(2, \mathbb{R})$ case we achieved this via the representation of P; here we analyze the representation of $H = AB$.

Identify $\widehat{B} \simeq \widehat{\mathbb{R}^{n-1}} \simeq \mathbb{R}^{n-1}$. By Proposition 2.5.3, we can write

$$\pi|_B \ \simeq\ \pi_{(\mu, \mathcal{H}_\lambda)} \qquad (\lambda \in \widehat{\mathbb{R}^{n-1}}),$$

where μ is a σ-finite Borel measure on $\widehat{\mathbb{R}^{n-1}}$ and $\mathcal{H}_\lambda$ is a measurable field of Hilbert spaces. By Proposition 2.5.4, the adjoint action of $a \in A$ on B induces the pushforward $a_*\mu$ on $\widehat{\mathbb{R}^{n-1}}$ and acts on fibers by $\mathcal{H}_\lambda \mapsto \mathcal{H}_{a^{-1}\lambda}$.

Since aba^{-1} scales the coordinates by $(a_1/a_j)\, b_j$ (for $j = 2, \dots, n$), the induced action on $\widehat{\mathbb{R}^{n-1}}$ is

$$(a \cdot \lambda)_j = \tfrac{a_1}{a_j}\, \lambda_j, \qquad j = 2, \dots, n.$$

Let $E, F \subseteq \widehat{\mathbb{R}^{n-1}}$ be compact sets disjoint from the coordinate hyperplanes $\{\lambda_j = 0\}$, $j = 2, \dots, n$. Then for $a \in A$ outside a sufficiently large compact set we have $(a \cdot E) \cap F = \emptyset$. Arguing exactly as in the proof of Theorem 2.5.6 (using Proposition 2.5.5 (2)), we deduce: if $\mu\big(\bigcup_{j=2}^{n}\{\lambda_j = 0\}\big) = 0$, then all matrix coefficients of $\pi|_A$ vanish at ∞. By Lemma 2.6.2, this implies the theorem.

Therefore, it remains to show that $\mu(\{\lambda_j = 0\}) > 0$ is impossible for each $j = 2, \dots, n$. Fix $i \in \{2, \dots, n\}$ and suppose $\mu(\{\lambda_i = 0\}) > 0$. Consider the subgroup

$$B_i = \{\, b \in B : \ b_j = 0 \text{ for all } j \neq i \,\},$$

which is isomorphic to $\mathbb{R}$. In the direct–integral model for $\pi|_B$, the subspace

$$\mathcal{H}_{\{\lambda_i = 0\}} = \int_{\{\lambda_i = 0\}}^{\oplus} \mathcal{H}_\lambda \, \mathrm{d}\mu(\lambda)$$

is non-zero and B_i-invariant (indeed, $\lambda_i = 0$ means B_i acts trivially on the fiber). Define the closed subgroup $H_i \leq G$ by

$$H_i = \left\{ \begin{pmatrix} \alpha & 0 & \cdots & \beta & \cdots & 0 \\ 0 & 1 & & 0 & & 0 \\ \vdots & & \ddots & \vdots & & \vdots \\ \gamma & 0 & \cdots & \delta & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} : \ \alpha\delta - \beta\gamma = 1 \right\} \simeq \mathrm{SL}(2, \mathbb{R}),$$

where the only possibly non-trivial entries outside the diagonal lie in the 2×2 block on rows/columns $\{1, i\}$ (all other off–diagonal entries are 0, and all remaining diagonal entries are 1). In particular, $B_i \leq H_i$ corresponds to $\alpha = \delta = 1$, $\gamma = 0$, $\beta \in \mathbb{R}$. Thus $B_i \leq H_i \leq G$ and B_i is non-compact in H_i.

Restrict π to H_i. We obtain a unitary representation of $H_i \simeq \mathrm{SL}(2, \mathbb{R})$ with a non-zero B_i-invariant vector. By Theorem 2.6.3 (applied inside H_i) and the fact that B_i is non-compact, this forces the existence of a non-zero H_i-invariant vector. In particular, the diagonal subgroup

$$A_i = H_i \cap A \simeq \big\{ \mathrm{diag}(a_1, \dots, a_n) : a_1 a_i = 1, \ a_j = 1 \ (j \neq 1, i) \big\}$$

has non-trivial invariant vectors.

Let

$$\mathcal{W} = \{\xi \in \mathcal{H} : \ \pi(a)\xi = \xi \text{ for all } a \in A_i\}$$

be the subspace of A_i-fixed vectors. It suffices to show that $\mathcal{W}$ is G-invariant. Indeed, the representation $\pi_{\mathcal{W}}$ of G on $\mathcal{W}$ has kernel containing A_i; by simplicity of $G = \mathrm{SL}(n, \mathbb{R})$, this forces $\ker \pi_{\mathcal{W}} = G$, so every vector in $\mathcal{W}$ is G-invariant, contradicting our assumptions.

We now prove G-invariance of $\mathcal{W}$. For $k \neq j$, let $B_{kj} \leq G$ be the one-dimensional unipotent subgroup

$$B_{kj} = \left\{ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & b & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} : \ b \in \mathbb{R} \right\},$$

where the only off–diagonal entry that may be non-zero is in row k, column j (all diagonal entries are 1). Consider two cases.

(a) If $k \notin \{1, i\}$ and $j \notin \{1, i\}$, then B_{kj} commutes with A_i, hence preserves $\mathcal{W}$.

(b) If $\{k, j\} \cap \{1, i\} \neq \emptyset$, then A_i normalizes B_{kj}. Indeed, writing $b_{kj} \in B_{kj}$ for the matrix with (k, j)-entry equal to $b \in \mathbb{R}$ and all other off–diagonal entries 0 (and 1's on the diagonal), and $a = \mathrm{diag}(a_1, \ldots, a_n) \in A_i$ (so $a_1 a_i = 1$ and $a_\ell = 1$ for $\ell \notin \{1, i\}$), we have

$$a\, b_{kj}\, a^{-1} = b'_{kj}, \qquad \text{with } \ (k, j)\text{-entry } b' = \frac{a_k}{a_j}\, b \in \mathbb{R}.$$

In particular, when $\{k, j\} = \{1, i\}$ we get

$$a\, b_{1i}\, a^{-1} = b'_{1i} \text{ with entry } b' = a_1^2 b, \qquad a\, b_{i1}\, a^{-1} = b'_{i1} \text{ with entry } b' = a_1^{-2} b,$$

so the 2–dimensional subgroup $A_i B_{1i}$ (resp. $A_i B_{i1}$) is isomorphic to $P = AN$ (resp. its opposite), via

$$A_i \ni \mathrm{diag}(a_1, 1, \ldots, a_1^{-1}, \ldots, 1) \longmapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \qquad b_{1i} \longmapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

By Corollary 2.5.7, every A_i–invariant vector is then B_{1i}– (or B_{i1}–) invariant. For the remaining possibilities with exactly one of k, j in $\{1, i\}$, choose a permutation matrix $p \in K$ sending that pair of indices to $\{1, i\}$. Then $\pi(p)\mathcal{W}$ is pA_ip^{-1}–invariant; applying the previous argument to the representation conjugated by p shows that $\pi(p)\mathcal{W}$ is $B_{p(k)p(j)}$–invariant. Conjugating back by p yields that $\mathcal{W}$ is B_{kj}–invariant. Hence in all cases B_{kj} preserves $\mathcal{W}$.

Finally, $A_i \leq A$ and A is Abelian, so it preserves $\mathcal{W}$. Since G is generated by A together with all the subgroups B_{kj}, it follows that $\mathcal{W}$ is G-invariant. This completes the proof. □

Finally, we sketch the proof of the general case, which goes very similarly as the above.

Proof. (Proof sketch of Theorem 2.4.8) Let G be as in the theorem. For definitions and basic facts about the following argument, check section A.8 of the Appendix. Fix a maximal $\mathbb{R}$–split torus $A \leq G$, and write $\mathfrak{a} = \mathrm{Lie}(A)$. We have the root space decomposition (A.8.7)

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \ [H, X] = \alpha(H)X \ \forall H \in \mathfrak{a}\},$$

where $\Sigma \subseteq \mathfrak{a}^* \setminus \{0\}$ is the root system.

As in Proposition A.8.13, there exists a semisimple G' such that $A \leq G' \leq G$, G' is closed in G, A is a maximal $\mathbb{R}$–split torus of G', and its Lie algebra is

$$\mathfrak{g}' \ = \ \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}'_\alpha,$$

where all the $\mathfrak{g}'_\alpha$ are one-dimensional.

As in Proposition A.8.14, choose a set $S \subseteq \Sigma$ of linearly independent positive roots such that $\mathfrak{a}^* = \mathrm{span}(S)$ and $\alpha + \beta \notin \Sigma$ for all $\alpha, \beta \in S$. The Lie subalgebra

$$\mathfrak{b} \ = \ \bigoplus_{\alpha\in S} \mathfrak{g}'_\alpha$$

is Abelian, so $B = \exp(\mathfrak{b}) \leq G'$ is an Abelian subgroup normalized by A; moreover, $\dim B = |S| = \dim A$. Finally, $\exp : \mathfrak{b} \to B$ is a diffeomorphism, hence we can identify $B \simeq \mathbb{R}^{\dim A}$. Indeed: first observe that for each $X \in \mathfrak{b}$, $\mathrm{ad}_{\mathfrak{g}'}(X)$ is nilpotent by construction of $\mathfrak{b}$ (write $X = \sum_{\alpha\in S} c_\alpha X_\alpha$ and note that each summand satisfies $[X_\alpha, \mathfrak{g}'_\beta] \subseteq \mathfrak{g}'_{\alpha+\beta}$ for fixed $\beta \in \Sigma$, so $\mathrm{ad}_{\mathfrak{g}'}(X)^n(\mathfrak{g}'_\beta)$ is zero for n large enough). Furthermore, $B \cap \mathrm{Z}(G') = \{e\}$, because any $z = \exp(X) \in B \cap \mathrm{Z}(G')$ satisfies $\mathrm{id} = \mathrm{Ad}_{G'}(\exp X) = \exp(\mathrm{ad}_{\mathfrak{g}'}(X))$, but this forces $\mathrm{ad}_{\mathfrak{g}'}(X) = 0$ by nilpotency of $\mathrm{ad}_{\mathfrak{g}'}(X)$, hence $X = 0$ by semisimplicity of $\mathfrak{g}'$, and $z = \exp(0) = e$. This lets us identify $B \simeq \mathrm{Ad}_{G'}(B)$ and $\mathfrak{b} \simeq \mathrm{ad}_{\mathfrak{g}'}(\mathfrak{b})$. But $\mathrm{ad}_{\mathfrak{g}'}(\mathfrak{b})$ is nilpotent, hence $\exp : \mathrm{ad}_{\mathfrak{g}'}(\mathfrak{b}) \to \mathrm{Ad}_{G'}(B)$ is a diffeomorphism.

The representation of AB can be analyzed exactly as in the $\mathrm{SL}(n, \mathbb{R})$ case by applying Proposition 2.5.3 and Proposition 2.5.4. As in the rank–one reductions (using the embedded $\mathrm{SL}(2, \mathbb{R})$'s generated by $\mathfrak{g}'_{\pm\alpha}$ and $\mathfrak{a}_\alpha = \mathbb{R}H_\alpha$), we obtain that either all matrix coefficients along A vanish at ∞, or there exists a one–parameter subgroup $A_0 \leq A$ and a non-zero vector fixed by A_0.

To justify the rank–one step uniformly, one may work in the universal covering $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ of the $\mathrm{SL}(2, \mathbb{R})$–subgroups if needed: for $N \leq \mathrm{SL}(2, \mathbb{R})$, its connected lift $\widetilde{N} \leq \widetilde{\mathrm{SL}(2, \mathbb{R})}$ is still isomorphic to N, and the argument proving that

A–invariant vectors are N–invariant (Corollary 2.5.7) carries over by lifting the $P = AN$–action and projecting back.

Finally, as in the $\mathrm{SL}(n, \mathbb{R})$ proof, set $\mathcal{W} = \{\xi : \pi(a)\xi = \xi \ \forall a \in A_0\}$. The root subgroups U_α either commute with A_0 or, together with A_0, generate a copy of $P_\alpha \simeq AN$ inside the corresponding $\mathrm{SL}(2, \mathbb{R})$; by Corollary 2.5.7 they preserve $\mathcal{W}$. Since G' is generated by A and the root subgroups U_α, it follows that $\mathcal{W}$ is G'–invariant. Now write $G = \prod_i G_i$ with each G_i simple and non-compact, and note that $A_0 \leq G_j$ for some factor G_j. Applying the previous argument inside G_j shows that $\mathcal{W}$ is G_j–invariant, hence $\pi|_{G_j}$ has a non-zero invariant vector, contradicting the hypothesis of the theorem. Therefore, only the first alternative can occur, and all matrix coefficients of π vanish at ∞. $\square$

Appendix A

Preliminaries

A.1 Measurable spaces

The following two sections are dedicated to collecting some definitions and basic facts about measure theory. For an extended presentation on the basics of measure theory, check [Coh13].

Definition A.1.1 (Measurable space) Let X be a set. A collection $\mathscr{B}$ of subsets of X is called a σ-algebra if it contains the empty set, is closed under complements, and closed under countable unions. In this case, $(X, \mathscr{B})$ is called a measurable space. If $\mathscr{A}$ is any family of subsets of X, the smallest σ-algebra containing $\mathscr{A}$ is called the σ-algebra generated by $\mathscr{A}$.

Remark A.1.2 Every topological space X is a measurable space with the Borel σ-algebra $\mathscr{B}$ generated by open sets. Any $B \in \mathscr{B}$ is called a Borel subset of X.

Definition A.1.3 (Measurable maps) A map $f : (X, \mathscr{B}) \to (Y, \mathscr{C})$ between measurable spaces is called measurable if $f^{-1}(C) \in \mathscr{B}$ for every $C \in \mathscr{C}$. It is called a (measurable) isomorphism if it is bijective and both f and f^{-1} are measurable.

We will introduce additional methods for defining a σ-algebra on a given set, as outlined in the following definition.

Definition A.1.4 (Constructions) (a) Let X be a set, $(Y, \mathscr{C})$ a measurable space, and $f : X \to Y$ a mapping. The pullback σ-algebra $f^*\mathscr{C}$ is the coarsest σ-algebra that makes f measurable, namely,

$$f^*\mathscr{C} = \{f^{-1}(C) : C \in \mathscr{C}\}.$$

In particular, whenever X is a subset of a measurable space $(Y, \mathscr{C})$, we define the subspace σ-algebra on X to be the pullback σ-algebra under the inclusion map $i : X \hookrightarrow Y$.

(b) Let $(X, \mathscr{B})$ be a measurable space, Y a set, and $f : X \to Y$ a mapping. The pushforward σ-algebra $f_*\mathscr{B}$ on Y is the finest σ-algebra that makes f measurable, that is,

$$f_*\mathscr{B} = \{C \subseteq Y : f^{-1}(C) \in \mathscr{B}\}.$$

In particular, whenever $\sim$ is an equivalence relation on a measurable space X, we define the quotient σ-algebra on $X/\sim$ to be the pushforward σ-algebra under the quotient map $p : X \to X/\sim$.

Definition A.1.5 Let X be a measurable space.

(a) X is called countably separated if there exists a countable family of measurable sets $\{A_i\}_i$ which separates points in the following sense: for any two distinct points $x, y \in X$, there exists an A_i such that $x \in A_i$ and $y \notin A_i$, or $x \notin A_i$ and $y \in A_i$.

(b) X is called countably generated if it is countably separated by a family $\{A_i\}_i$ which also generates the σ-algebra.

Remark A.1.6 (1) Measurable subsets of countably separated (resp. generated) spaces are countably separated (resp. generated). One can check this by intersecting the family of separating sets with the measurable subset.

(2) Any second countable T_0 topological space is countably generated. Indeed, the countable basis for the topology (which generates the Borel σ-algebra) separates points: if $x \neq y$, then there exists an open set U containing x and not y (or vice versa). Then, there exists a basis element $B \subseteq U$, which is a set separating x and y.

(3) Hence, any second countable Hausdorff topological space is countably generated.

(4) Any separable metrizable space (which is therefore second countable and Hausdorff) is countably generated.

Proposition A.1.7 *(1) X is countably separated if and only if there exists an injective measurable map $X \to [0, 1]$.*

(2) X is countably generated if and only if X is measurably isomorphic to a subset of $[0, 1]$.

Proof. Let $\Omega = \{0, 1\}^{\mathbb{N}}$ with the product σ-algebra, and define $f : X \to \Omega$ by $f(x)_i = \chi_{A_i}(x)$ for $i \in \mathbb{N}$. Since each $\chi_{A_i} : X \to \{0, 1\}$ is measurable and the product σ-algebra is the smallest making all coordinate projections measurable, f is measurable.

Because $\{A_i\}$ separates points, for $x \neq y$ there exists i with $\chi_{A_i}(x) \neq \chi_{A_i}(y)$, hence $f(x) \neq f(y)$. Thus f is injective.

Let $\pi_i : \Omega \to \{0,1\}$ be the i-th coordinate projection. Then $\pi_i^{-1}(\{1\})$ is measurable in Ω, and

$$f^{-1}\big(\pi_i^{-1}(\{1\})\big) = A_i.$$

Intersecting with $f(X)$ gives

$$f(A_i) = f(X) \cap \pi_i^{-1}(\{1\}),$$

so $f(A_i)$ is measurable in the subspace $(f(X), \mathcal{B}(\Omega)|_{f(X)})$. This proves the claim. □

Definition A.1.8 (Standard measurable space) A measurable space is called standard if it is isomorphic to a Borel subset of a complete separable metric space.

Remark A.1.9 Standard measurable spaces are countably generated, by the remarks above.

Theorem A.1.10 *Any standard measurable space is either finite, isomorphic to* $\mathbb{Z}$*, or isomorphic to* $[0,1]$.

Theorem A.1.11 *If X is a standard measurable G-space, where G is a locally compact second countable group, and the action $G \curvearrowright X$ is smooth (i.e. X/G is countably separated), then X/G is standard and there exists a measurable section $\varphi : X/G \to X$ of the natural projection $p : X \to X/G$.*

Corollary A.1.12 *If $H \leq G$ is a closed subgroup of a locally compact second countable group G, then there is a measurable section $G/H \to G$ of the natural projection $G \to G/H$.*

A.2 Measures

Definition A.2.1 (Measure) A measure on a measurable space $(X, \mathscr{B})$ is a function $\mu : \mathscr{B} \to [0, \infty]$ which is countably additive and such that $\mu(\emptyset) = 0$.

We call μ a probability measure if $\mu(X) = 1$.

Sets with measure 0 (under μ) are called (μ-)null sets, and sets whose complement is null are called (μ-)conull sets.

Definition A.2.2 A measure on X is called σ-finite if there exists a countable collection $\{A_i\}_i$ of measurable sets with finite measure such that $\bigcup_i A_i = X$.

Definition A.2.3 (Absolute continuity and equivalence of measures) Let $(X, \mathscr{B})$ be a measurable space, and let μ, ν be two measures on X.

(a) We say that μ is absolutely continuous with respect to ν (write $\mu \ll \nu$) if every ν-null set is also a μ-null set.

(b) The measures μ and ν are said to be equivalent (write $\mu \sim \nu$) if $\mu \ll \nu$ and $\nu \ll \mu$. Equivalently, if they have the same null sets. This is an equivalence relation on the set of measures on $(X, \mathscr{B})$. A class under this equivalence relation is called a measure class.

Remark A.2.4 Every σ-finite measure μ is equivalent to a probability measure. Indeed, if $\{A_i\}_{i=1}^{\infty}$ is a family of sets of finite measure whose union is X, it is easily verified that

$$\nu : B \mapsto \nu(B) = \sum_{i=1}^{\infty} 2^{-i} \frac{\mu(B \cap A_i)}{\mu(A_i)}$$

is a probability measure on X equivalent to μ.

Definition A.2.5 (Borel measure) A Borel measure on a topological space X is a measure on the Borel σ-algebra of X.

Definition A.2.6 (Radon measure) A Radon measure on a topological space X is a Borel measure μ which is

(a) finite on compact sets,

(b) outer regular on Borel sets: for any Borel set B,

$$\mu(B) = \inf\{\mu(U) : U \supseteq B, U \text{ open}\}, \text{ and}$$

(c) inner regular on open sets: for any open set U,

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}.$$

We say that μ is regular if, additionally it is inner regular on all Borel sets.

Radon measures are important because they correspond to positive linear functionals on the space of continuous functions with compact support on a locally compact Hausdorff topological space X. This makes it possible to develop measure and integration from the point of view of functional analysis. The following theorem makes this precise.

Theorem A.2.7 (Riesz-Markov-Kakutani Representation Theorem) *Let X be a locally compact Hausdorff topological space. If Λ is a positive real linear functional on $\mathrm{C_c}(X)$ (that is, $\Lambda(f) \geq 0$ for all $f \in \mathrm{C_c}(X)$ with $f \geq 0$), then there exists a unique Radon measure μ on X that represents Λ, that is,*

$$\Lambda(f) = \int_X f \, d\mu$$

for all $f \in \mathrm{C_c}(X)$.

Obviously, the converse is true: every Radon measure on X gives rise to a positive linear functional on $\mathrm{C_c}(X)$ in this way. The theorem therefore establishes a one-to-one correspondence between Radon measures and positive linear functionals on $\mathrm{C_c}(X)$.

Remark A.2.8 Note that the theorem produces a Radon measure, which by definition is only inner regular on open sets. Fortunately, under mild conditions, the Radon measures arising from this representation are actually inner regular on all Borel sets. For instance, any σ-finite Radon measure on a locally compact Hausdorff space is inner regular on all Borel sets.

We close this section with a useful fact about countably generated spaces with σ-finite measure.

Theorem A.2.9 ([Coh13]) *Let $(S, \mathscr{B}, \mu)$ be a countably generated σ-finite measure space. Then, the space $\mathrm{L}^p(S, \mathscr{B}, \mu)$ is separable for all $1 \leq p < \infty$.*

A.3 Measures and topological groups

In this section, G is a locally compact, Hausdorff topological group.

Definition A.3.1 (Haar measure) A (left) (resp. right) Haar measure on G is a nonzero Radon measure μ on G which satisfies $\mu(gA) = \mu(A)$ (resp. $\mu(Ag) = \mu(A)$) for all $g \in G$ and all Borel sets $A \subseteq G$.

In virtue of the Riesz-Markov-Kakutani Representation Theorem, the Haar measure is equivalent to a positive linear functional on $\mathrm{C_c}(G)$. We hence use the notation $\mu(f) = \int_G f \mathrm{d}\mu = \int_G f(x) \mathrm{d}\mu(x)$ according to what we want to emphasize.

Theorem A.3.2 ([Fol16]) *A left (resp. right) Haar measure on G exists and is unique up to positive multiplicative constants.*

Proposition A.3.3 *Open sets have positive Haar measure.*

Proof. Since $m(G) > 0$, by inner regularity, there exists a compact set $K \subseteq G$ with $m(K) > 0$. For any open set $U \subseteq G$, we have that K can be covered by finitely many translates of U. Therefore, the measure of U cannot be 0. □

Proposition A.3.4 *Countable sets in non-discrete topological groups have Haar measure 0.*

Proof. Let G be a non-discrete group, m its Haar measure, and A a countable set. A is measurable, since it is the countable union of its points, and points in a Hausdorff space are closed, hence measurable. Since $m(A) = \sum_{x \in A} m(\{x\})$, we only need to show that points (singletons) in G have measure 0. Note that, by G-invariance of the Haar measure, all singletons have the same measure.

Recall that the Haar measure of any compact subset of G is finite. If we manage to find an infinite compact subset K of G, we would have that $\infty > m(K) \geq \sum_{x \in \text{countable} \subseteq K} m(\{x\})$ —where the sum is taken over any countably infinite subset of K—, making $m(\{x\}) = 0$.

Suppose no infinite compact subset exists. Since G is non-discrete, let $g \in G$ be a point which is not isolated. Then, by local compactness, it has a compact neighborhood V, which is finite by assumption, say $V = \{g, x_1, \ldots, x_n\}$. Now, since G is Hausdorff, we can find open neighborhoods W_i of g that do not contain x_i for each $i = 1, \ldots, n$. Finally, the finite intersection of neighborhoods

$$V \cap \left(\bigcap_{i=1}^{n} W_i \right) = \{g\}$$

is a neighborhood of g, contradicting the fact that g was not isolated. □

Proposition A.3.5 *Let G be compact. Then, any measurable automorphism $\varphi : G \to G$ preserves the Haar (probability) measure μ.*

Proof. The measure $\varphi_*\mu$ defined by $\varphi_*\mu(A) = \mu(\varphi^{-1}(A))$ is also a Haar measure, hence it is a constant multiple of μ. But $\varphi_*\mu(G) = \mu(\varphi^{-1}(G)) = \mu(G) = 1$, so $\varphi_*\mu = \mu$. □

(A.3.6) The modular function. The group G acts on $\mathrm{C_c}(G)$ on the left by conjugation on the argument, that is:

$$(g \cdot f)(x) = f(g^{-1}xg),$$

for $g \in G$ and $f \in \mathrm{C_c}(G)$. If μ is a Haar measure on G, one can easily verify that, given $g \in G$, the linear functional

$$f \mapsto \mu(g \cdot f)$$

is also a left Haar measure on G. Hence, there exists a positive constant $\Delta_G(g)$ such that

$$\mu(g \cdot f) = \Delta_G(g)\mu(f).$$

We call

$$\Delta_G : G \to \mathbb{R}_{>0}, \quad g \mapsto \Delta_G(g)$$

the modular function of G. It is a continuous homomorphism from G to the multiplicative group of positive real numbers.

Definition A.3.7 (Unimodular group) G is called unimodular if $\Delta_G \equiv 1$.

Examples A.3.8 (1) Any Abelian group is unimodular.

(2) Any compact group is unimodular, since there are no nontrivial compact subgroups of $(\mathbb{R}_{>0}, \cdot)$.

(3) Any discrete group is unimodular, since the Haar measure is the counting measure.

(4) Any connected semisimple Lie group G is unimodular, since $G = [G, G]$. This in turn implies that G does not admit a nontrivial homomorphism to $(\mathbb{R}_{>0}, \cdot)$.

Unimodularity is important, among other things, because of the following theorem.

Theorem A.3.9 (Weil formula, [Fol16]) *Suppose G is a locally compact Hausdorff group and $H \leq G$ is a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, μ is unique up to a constant factor, and this factor can be chosen so that we have*

$$\int_G f(x)dx = \int_{G/H} \left(\int_H f(x\xi)d\xi \right) d\mu(xH)$$

for $f \in \mathrm{C_c}(G)$.

In particular, if G and H are unimodular, there is a unique (up to scaling) G-invariant Radon measure on G/H.

The broader result —when we only assume that G is locally compact Hausdorff and H is closed— is mentioned in example 1.1.6 (1) of this text. For an ample treatment, see [Fol16, §2.6]. We summarize everything in the following theorem.

Theorem A.3.10 *Let G be a locally compact Hausdorff group and $H \leq G$ be a closed subgroup. Then, there exists a G-quasi-invariant Radon measure on G/H and any two such measures are equivalent.*

One instance in which the discussion about invariant measures is relevant is the following.

Definition A.3.11 (Lattice subgroup) A subgroup $\Gamma \leq G$ is called a lattice if it is discrete and there exists a finite G-invariant Radon measure on G/Γ.

Remark A.3.12 Note that, by theorem A.3.9, the existence of a G-invariant Radon measure on G/Γ implies that $\Delta_G|_\Gamma = \Delta_\Gamma \equiv 1$, since Γ, being discrete, is unimodular. However, more can be said about G:

Proposition A.3.13 *If G admits a lattice Γ, then G is unimodular.*

Proof. Since Γ is discrete and unimodular, we have $\Delta_G|_\Gamma \equiv 1$, so $\Gamma \subseteq \ker(\Delta_G)$. The modular function thus descends to a well-defined map $\Delta : G/\Gamma \to \mathbb{R}_{>0}$ satisfying $\Delta(gx) = \Delta_G(g)\Delta(x)$ for $g \in G$ and $x \in G/\Gamma$. Pushing forward the finite G-invariant measure on G/Γ via Δ yields a finite $\Delta_G(G)$-invariant measure on $\mathbb{R}_{>0}$. This is impossible unless $\Delta_G(G) = \{1\}$. □

A.4 Technical remarks

As we stated earlier (remark A.2.8), any σ-finite Radon measure is inner regular on all Borel sets. This implies:

Corollary A.4.1 *Let G be a second-countable locally compact Hausdorff group. Then the Haar measure on G is regular.*

(A.4.2) Polish spaces and regularity. By a Polish space, we mean a separable completely metrizable space. In Polish spaces, we have the following regularity result:

Proposition A.4.3 *Any finite Borel measure on a Polish space is regular.*

In particular, since locally compact Hausdorff second countable spaces are Polish, any finite Borel measure on a locally compact Hausdorff second countable space is regular.

Remark A.4.4 Recall that any σ-finite measure is equivalent to a probability measure (remark A.2.4). Therefore, any σ-finite Borel measure on a Polish space is equivalent to a Borel probability measure, which is regular by Proposition A.4.3. This lets us state the following corollary.

Corollary A.4.5 *Let G be a locally compact Hausdorff second countable group and $H \leq G$ be a closed subgroup. There exists G-quasi-invariant σ-finite measure on G/H, and any two such measures are equivalent. We summarize this by saying that there exists a unique G-invariant measure class on G/H.*

Proof. Existence follows from Theorem A.3.10. For uniqueness, any two G-quasi-invariant measures μ and ν on G/H are equivalent to G-quasi-invariant probability measures $\tilde{\mu}$ and $\tilde{\nu}$ on G/H. Since G/H is locally compact, Hausdorff, and second countable, these are regular by Proposition A.4.3. Therefore, again by A.3.10, $\tilde{\mu}$ and $\tilde{\nu}$ are equivalent, and so are μ and ν. □

A.5 Unitary representations

Let $\mathcal{H}$ be a Hilbert space. The inner product on $\mathcal{H}$ is denoted by $\langle \xi, \eta \rangle$ and is assumed to be linear in the first variable. We say that a linear operator $U : \mathcal{H} \to \mathcal{H}$ is *unitary* if it is onto and preserves the inner product, that is, $\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. This, of course, implies boundedness of U. The group of all unitary operators on $\mathcal{H}$ is denoted by $\mathcal{U}(\mathcal{H})$, which is always endowed with the strong operator topology (namely, the topology of pointwise convergence on $\mathcal{H}$).

Definition A.5.1 (Unitary representation) A unitary representation of a locally compact Hausdorff group G in a Hilbert space $\mathcal{H}$ is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ that is continuous (in the strong operator topology).

Continuity in the strong operator topology means that the map

$$G \to \mathcal{H}, \quad g \mapsto \pi(g)\xi$$

is continuous for all $\xi \in \mathcal{H}$. It is worth noting that strong continuity is implied by the (apparently) weaker condition of weak continuity, that is, the condition that the map

$$G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle \tag{A.1}$$

is continuous for all $\xi, \eta \in \mathcal{H}$. This is because, on $\mathcal{U}(\mathcal{H})$, the strong operator topology and the weak operator topology coincide. The map (A.1) deserves its own name:

Definition A.5.2 (Matrix coefficient) Given $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation, and $\xi, \eta \in \mathcal{H}$, the map

$$G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is called a matrix coefficient of π.

The notion of equivalence of unitary representations is defined as follows.

Definition A.5.3 (Unitary equivalence) Two unitary representations $\pi_1 : G \to \mathcal{U}(\mathcal{H}_1)$ and $\pi_2 : G \to \mathcal{U}(\mathcal{H}_2)$ are said to be unitarily equivalent if there exists an isometric surjective operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$. We write $\pi_1 \simeq \pi_2$.

Unitary representations are likely to be found when studying the action of a group G on a space S, as the following Proposition shows.

Proposition A.5.4 ([BdlHV08]) *Let G be a locally compact, σ-compact group, and (S, μ) a σ-finite G-space such that μ is invariant and such that $\mathrm{L}^2(S, \mu)$ is separable. Then, the map*

$$(\pi(g)f)(s) = f(g^{-1}s), \quad g \in G, s \in S,$$

defines a unitary representation $\pi : G \to \mathcal{U}(\mathrm{L}^2(S, \mu))$ of G in $\mathrm{L}^2(S, \mu) \equiv \mathrm{L}^2(S)$.

Example A.5.5 If a locally compact Hausdorff second countable group G acts on a standard measurable space S with G-invariant measure μ, then

$$(\pi(g)f)(s) = f(g^{-1}s), \quad g \in G, s \in S,$$

defines a unitary representation $\pi : G \to \mathcal{U}(\mathrm{L}^2(S))$. This is because of Theorem A.2.9 and Proposition A.5.4.

A.6 Character theory

The theory described in this section is usually called character theory or Pontryagin duality. For the most part, we list some results following [EW11, §C.3]. A more ample (and accessible) treatment can be found in [Fol16, Ch. 4].

Character theory is a rather powerful theory generalizing Fourier analysis on the circumference S^1 to locally compact Abelian (LCA) groups. Throughout the section, G will denote a LCA group.

(A.6.1) Characters and the dual group. A character on G is a continuous homomorphism

$$\chi : G \to \mathrm{S}^1 = \{z \in \mathbb{C} : |z| = 1\}^1.$$

We denote by $\widehat{G}$ the set of characters on G. Note that $\widehat{G}$ is an Abelian group under pointwise multiplication of characters, namely

$$(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g), g \in G.$$

It is also usual to write $\langle g, \chi\rangle = \chi(g)$ to emphasize that it is a pairing between G and $\widehat{G}$. We call $\widehat{G}$ the dual group to G.

Theorem A.6.2 *For any compact Abelian group G, the set of characters forms a Hilbert basis for $L^2(G)$.*

Proposition A.6.3 *Let $X = \{\pm1\}^{\mathbb{Z}_{>0}} = \prod_1^\infty \{\pm1\}$. Then, the dual group $\widehat{X}$ consists of all functions of the form $p_{i_1} \cdots p_{i_n}$, where $p_i : X \to \{\pm1\} \subseteq \mathrm{S}^1$ is the projection on the i-th factor and $i_1, \ldots, i_n$ is a (possibly empty) finite sequence of positive integers without repetitions.*

Proof. Let $\chi : X \to \mathrm{S}^1$ be an arbitrary character. Since χ is a homomorphism and $x^2 = \mathrm{id}$ for every $x \in X$, we have that $\chi(x)^2 = 1$, that is, $\chi(x) \in \{\pm1\}$.

On the other hand, since χ is continuous, we claim that it can only depend on a finite number of coordinates. That is, there exists $N \in \mathbb{Z}_{>0}$ such that if $x_k = y_k$ for all $k \leq N$, then $\chi((x_k)_k) = \chi((y_k)_k)$.

This claim follows from the fact that X is a compact metrizable space with metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \left(\frac{1}{2}\right)^{N(x,y)}, & \text{otherwise,} \end{cases}$$

where $N(x, y) = \min\{k \in \mathbb{Z}_{>0} : x_k \neq y_k\}$. Since X is compact, χ is uniformly continuous, so there exists a $\delta > 0$ such that if $d(x, y) < \delta$, then $\chi(x) = \chi(y)$. This assertion is equivalent to the claim.

[1]Since $\mathrm{S}^1 = \mathcal{U}(\mathbb{C})$, a character is no more than a 1-dimensional unitary representation.

From the previous fact, we deduce that χ factors through the projection $\pi_N : X \to \{\pm 1\}^N$ on the first N coordinates:

$$X \xrightarrow{\pi_N} \{\pm 1\}^N \xrightarrow{f} \{\pm 1\}.$$

It is immediate to see that any such homomorphism f is the multiplication of projections $p_{i_1} \cdots p_{i_n}$ for some sequence of distinct integers $1 \leq i_1, \ldots, i_n \leq N$. This is what we wanted to prove. □

A.7 Direct integrals

This is a summary of the exposition in [Zim84, §2.3] and [BdlHV08, §F.5].

(A.7.1) Fields of Hilbert spaces. Suppose (M, μ) is a measure space and that for each $x \in M$ we have a Hilbert space $\mathcal{H}_x$ such that the assignment $x \mapsto \mathcal{H}_x$ is piecewise constant, namely, that there is a disjoint decomposition of M into measurable sets,

$$M = \bigsqcup_{i=1}^{\infty} M_i,$$

such that for $x, y \in M_i$, $\mathcal{H}_x = \mathcal{H}_y$. We call this a *field of Hilbert spaces over* M. By a *section* (or a *vector field*) of $(\mathcal{H}_x)_{x \in M}$ we mean an assignment

$$M \ni x \mapsto f(x) \in \mathcal{H}_x.$$

Since $\mathcal{H}_x$ is piecewise constant, the notion of measurability for f is easily defined, namely, that it be a measurable function on each M_i into the corresponding Hilbert space[2].

(A.7.2) Direct integral of a field. Let $\mathrm{L}^2(M, \mu, (\mathcal{H}_x)_{x \in M})$ be the set of sections f such that $\int_M \|f\|^2 \, \mathrm{d}\mu < \infty$, identifying two sections if they coincide μ-almost everywhere. The space of square-integrable sections is also denoted by

$$\mathcal{H} = \int_M^{\oplus} \mathcal{H}_x := \mathrm{L}^2(M, \mu, (\mathcal{H}_x)_{x \in M}),$$

and is called the *direct integral* of the field $(\mathcal{H}_x)_{x \in M}$. It is also a Hilbert space under the inner product

$$\langle f, g \rangle = \int_M \langle f(x), g(x) \rangle \, \mathrm{d}\mu(x).$$

[2]Here, the Borel structure on $\mathcal{H}_x$ is the one induced by the weak topology, which (for any Hilbert space) coincides with the one induced by the norm topology (see [Edg79]). This is not the case for general Banach spaces (see [Tal78]).

Examples A.7.3 (1) Let M be a countable set and let μ be a measure on M such that $\mu(\{x\}) \neq 0$ for all $x \in M$. Then every section is measurable and

$$\int_M^{\oplus} \mathcal{H}_x \, \mathrm{d}\mu(x) = \bigoplus_{x \in M} \mathcal{H}_x$$

is the direct sum of the Hilbert spaces $\mathcal{H}_x$, $x \in M$.

(2) Let (M, μ) be a σ-finite measure space and let $\mathcal{H}_x = \mathbb{C}$ for all $x \in M$. The measurable sections are precisely the measurable complex-valued functions on M. Then

$$\int_M^{\oplus} \mathcal{H}_x \, \mathrm{d}\mu(x) = L^2(M, \mu).$$

(A.7.4) Direct integral of a field of unitary representations. Suppose now that (M, μ) is σ-finite and that for each $x \in M$ we have a unitary representation $\pi_x : G \to \mathcal{H}_x$ for a locally compact Hausdorff second countable group G and separable Hilbert spaces $\mathcal{H}_x$. Similarly as before, we say that $x \mapsto \pi_x$ is a *measurable field of unitary representations* if $(x, g) \mapsto \pi_x(g)$ is a measurable function on each $M_i \times G$. We can then define a new representation of G on $\mathcal{H} = \int_M^{\oplus} \mathcal{H}_x$:

$$\pi = \int_M^{\oplus} \pi_x, \quad (\pi(g)f)(x) = (\pi_x(g))(f(x)),$$

called the *direct integral* of the field $(\pi_x)_{x \in M}$.

Examples A.7.5 If M is countable and $\mu(\{x\}) \neq 0$ for all $x \in M$, then π is just the direct sum $\bigoplus_{x \in M} \pi_x$ on $\bigoplus_{x \in M} \mathcal{H}_x$.

A basic result on direct integrals of unitary representations is the following.

Proposition A.7.6 (Direct integral decomposition) *Any unitary representation π of a locally compact Hausdorff second countable group G on a separable Hilbert space is unitarily equivalent to one the form $\int_M^{\oplus} \pi_x$ for some standard measure space M with finite measure μ, where all π_x are irreducible.*

This proposition is very useful because it allows to reduce many questions about an arbitrary unitary representation to the case of an irreducible representation. For instance:

Proposition A.7.7 *Let $\pi = \int_M^{\oplus} \pi_x$.*

(1) Suppose all matrix coefficients of all π_x vanish at ∞. Then all matrix coefficients of π vanish at ∞.

(2) π has a non-trivial invariant vector if and only if for x in a set of positive measure, π_x has a non-trivial invariant vector.

A.8 Roots in semisimple Lie algebras

This section is dedicated to describing briefly the theory of roots in semisimple real Lie algebras, which is needed for the general proof of Moore's theorem. For a complete treatment, we recommend [Hel78].

Throughout, $\mathfrak{g}$ will denote a real semisimple Lie algebra, and $\mathrm{B}_{\mathfrak{g}}$ its Killing form.

Definition A.8.1 (Cartan involution) A Cartan involution is an automorphism $\Theta : \mathfrak{g} \to \mathfrak{g}$ such that $\Theta^2 = \mathrm{id}_{\mathfrak{g}}$ and the bilinear form

$$\langle X, Y \rangle_\Theta = -\mathrm{B}_{\mathfrak{g}}(X, \Theta Y), \quad X, Y \in \mathfrak{g}$$

is positive definite.

Proposition A.8.2 *Any real semisimple Lie algebra has a Cartan involution.*

(A.8.3) Cartan decomposition. Let Θ be a Cartan involution, which will be fixed from now on. Then, $\mathfrak{g}$ decomposes as a direct sum of the eigenspaces of Θ:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is eigenspace of Θ with eigenvalue 1 and $\mathfrak{p}$ is eigenspace of Θ with eigenvalue -1. Notice that $\mathfrak{k}$ is a subalgebra, while $\mathfrak{p}$ is not. This decomposition is called the *Cartan decomposition* associated to Θ.

Lemma A.8.4 *If* $X \in \mathfrak{p}$*, then* $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathrm{End}(\mathfrak{g})$ *is* $\langle \cdot, \cdot \rangle_\Theta$*-self-adjoint.*

Proof. Take $Y, Z \in \mathfrak{g}$. Then,

$$\begin{aligned} \langle \mathrm{ad}_{\mathfrak{g}}(X)Y, Z \rangle_\Theta &= \langle [X, Y], Z \rangle_\Theta = -\mathrm{B}_{\mathfrak{g}}([X, Y], \Theta Z) = \mathrm{B}_{\mathfrak{g}}(Y, [X, \Theta Z]) \\ &= \mathrm{B}_{\mathfrak{g}}(Y, -\Theta[X, Z]) = -\mathrm{B}_{\mathfrak{g}}(Y, \Theta[X, Z]) = \langle Y, [X, Z] \rangle_\Theta \\ &= \langle Y, \mathrm{ad}_{\mathfrak{g}}(X)Z \rangle_\Theta. \end{aligned}$$

□

As a consequence of this lemma, if $\mathfrak{a} \subseteq \mathfrak{p}$ is Abelian, then $\{\mathrm{ad}_{\mathfrak{g}}(H) : H \in \mathfrak{a}\}$ is a family of mutually commuting self-adjoint endomorphisms of $\mathfrak{g}$, hence simultaneously orthogonally diagonalizable. This motivates the following definition.

Definition A.8.5 (Roots) Let $\mathfrak{a} \subseteq \mathfrak{p}$ be an Abelian subalgebra. Let $\alpha \in \mathfrak{a}^*$. Define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}_{\mathfrak{g}}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\alpha \neq \{0\}$, we call α a root of $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_\alpha$ a root space.

We immediately have the following:

Proposition A.8.6 *(1)* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ *for all* $\alpha, \beta \in \mathfrak{a}^*$.

(2) $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ *for all* $\alpha \in \mathfrak{a}^*$*, and* $\Theta|_{\mathfrak{g}_\alpha} : \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ *is an isomorphism.*

(A.8.7) Root space decomposition. Observe that $\mathfrak{a} \subseteq \mathfrak{g}_0 = \mathrm{Centr}_{\mathfrak{g}}(\mathfrak{a})$. Let Σ denote the set of nonzero roots of $(\mathfrak{g}, \mathfrak{a})$. Then, by the fact that $\{\mathrm{ad}_{\mathfrak{g}}(H) : H \in \mathfrak{a}\}$ is a family of mutually commuting self-adjoint endomorphisms of $\mathfrak{g}$, we have a $\langle\cdot,\cdot\rangle_\Theta$-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

It also follows from the finite dimensionality of $\mathfrak{g}$ that Σ is finite.

The case where $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal among all Abelian subalgebras of $\mathfrak{p}$ is particularly interesting, and we will assume it from now on. In this case, we also have the following definition.

Definition A.8.8 (Maximal $\mathbb{R}$-split torus) Let G be a Lie group with Lie algebra $\mathfrak{g}$. For a fixed Cartan involution Θ and $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal Abelian subalgebra, a maximal $\mathbb{R}$-split torus of G is the connected Lie subgroup A with Lie algebra $\mathfrak{a}$.

A final remark on terminology:

Remark A.8.9 Since $\mathrm{B}_{\mathfrak{g}}$ is an inner product on $\mathfrak{a}$, we get for each $\alpha \in \Sigma$ a unique $H_\alpha \in \mathfrak{a}$ that represents it:

$$\alpha = \mathrm{B}_{\mathfrak{g}}(\cdot, H_\alpha).$$

Theorem A.8.10 (Σ is a root system) *Let* $\mathfrak{a} \subseteq \mathfrak{p}$ *be a maximal Abelian subalgebra of* $\mathfrak{p}$*, and* $\Sigma \subseteq \mathfrak{a}^* \setminus \{0\}$ *the set of nonzero roots of* $(\mathfrak{g}, \mathfrak{a})$*. Then,* Σ *is a root system, meaning that*

(1) Σ *spans* $\mathfrak{a}^*$.

(2) For all $\alpha, \beta \in \Sigma$,

$$\beta - \frac{2\mathrm{B}_{\mathfrak{g}}(H_\alpha, H_\beta)}{\mathrm{B}_{\mathfrak{g}}(H_\alpha, H_\alpha)}\alpha \in \Sigma.$$

(3) For all $\alpha, \beta \in \Sigma$,

$$\frac{2\mathrm{B}_{\mathfrak{g}}(H_\alpha, H_\beta)}{\mathrm{B}_{\mathfrak{g}}(H_\alpha, H_\alpha)} \in \mathbb{Z}.$$

An interesting fact about root spaces is that they give a way to construct copies of $\mathfrak{sl}(2, \mathbb{R})$ inside $\mathfrak{g}$. More particularly, recall that

$$\mathfrak{sl}(2,\mathbb{R}) = \mathrm{span}\left\{ \mathrm{e}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathrm{e}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

with commutation relations $[\mathrm{e}_+, \mathrm{e}_-] = h$, $[h, \mathrm{e}_\pm] = \pm 2\mathrm{e}_\pm$. This construction is a consequence of the following lemma.

Lemma A.8.11 *Let $\alpha \in \Sigma$ and $X \in \mathfrak{g}_\alpha \setminus \{0\}$. Let x_α be the unique positive multiple of X such that*

$$\langle x_\alpha, x_\alpha \rangle_\Theta = \frac{2}{\mathrm{B}_\mathfrak{g}(H_\alpha, H_\alpha)}.$$

Let $y_\alpha = -\Theta(x_\alpha)$, and $h_\alpha = \frac{2H_\alpha}{\mathrm{B}_\mathfrak{g}(H_\alpha, H_\alpha)}$. Then,

$$[x_\alpha, y_\alpha] = h_\alpha, \quad [h_\alpha, x_\alpha] = 2x_\alpha, \quad [h_\alpha, y_\alpha] = -2y_\alpha.$$

Corollary A.8.12 *Let $\alpha \in \Sigma$, $X \in \mathfrak{g}_\alpha \setminus \{0\}$, $x_\alpha, y_\alpha, h_\alpha$ be as in the previous lemma. Then, the linear map*

$$\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}, \quad e_+ \mapsto x_\alpha, \quad e_- \mapsto y_\alpha, \quad h \mapsto h_\alpha,$$

is an injective Lie algebra homomorphism with image

$$\mathfrak{sl}(2, \mathbb{R})_X = \operatorname{span}\{x_\alpha, y_\alpha, h_\alpha\}.$$

We call $(x_\alpha, y_\alpha, h_\alpha)$ a $\mathfrak{sl}(2, \mathbb{R})$-triple. Choose a triple for every α, $\{(x_\alpha, y_\alpha, h_\alpha) : \alpha \in \Sigma\}$, such that $x_{-\alpha} = y_\alpha$ for all α. In this case, call $\mathfrak{g}'_\alpha = \mathbb{R}x_\alpha \leq \mathfrak{g}_\alpha$. Then we have the following proposition.

Proposition A.8.13 *The space*

$$\mathfrak{g}' = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}'_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$ and it is semisimple. Moreover, if $\mathfrak{g} = \mathrm{Lie}(G)$ for some semisimple Lie group G, then the connected Lie subgroup G' corresponding to $\mathfrak{g}'$ is closed in G.

Proof. (Sketch of proof) Lie subalgebra: From the construction, it is clear that $[\mathfrak{a}, \mathfrak{a}] = 0$, $[\mathfrak{a}, \mathfrak{g}'_\alpha] \subseteq \mathfrak{g}'_\alpha$, $[\mathfrak{g}'_\alpha, \mathfrak{g}'_\beta] \subseteq \mathfrak{g}'_{\alpha+\beta}$ (or 0), and $[\mathfrak{g}'_\alpha, \mathfrak{g}'_{-\alpha}] \subseteq \mathfrak{a}$. Therefore $\mathfrak{g}'$ is closed under brackets.

Semisimple: Let $\mathfrak{r}'$ be the solvable radical of $\mathfrak{g}'$. Then, $\mathfrak{r}' \subseteq \mathfrak{a}$ (if not, some $x_\alpha \in \mathfrak{r}'$, then $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{r}'$, and also $-2y_\alpha \in \mathfrak{r}'$, yielding a copy of $\mathfrak{sl}(2, \mathbb{R})$ inside $\mathfrak{r}'$). Therefore, $[\mathfrak{r}', \mathfrak{g}'_\alpha] \subseteq \mathfrak{r}' \cap \mathfrak{g}'_\alpha = 0$ and $[\mathfrak{r}', \mathfrak{a}] = 0$, so $\mathfrak{r}' \subseteq \mathrm{Z}(\mathfrak{g}')$. Now, for $Z \in \mathfrak{r}'$ and $\alpha \in \Sigma$, $0 = [Z, x_\alpha] = \alpha(Z)x_\alpha$, so $\alpha(Z) = 0$ for all $\alpha \in \Sigma$. Since Σ spans $\mathfrak{a}^*$, we have $Z = 0$.

Closedness of G': any connected semisimple Lie subgroup of a semisimple group is closed (see [ora15]). ☐

We end this section with a small proposition that is also useful in the general proof of Moore's theorem.

Proposition A.8.14 *Given $H \in \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker \alpha$, there exists $S \subseteq \Sigma$ such that*

(1) S is a basis of $\mathfrak{a}^*$.

(2) $S \subseteq \Sigma^+(H) = \{\alpha \in \Sigma : \alpha(H) > 0\}$ (S consists of H-positive roots).

(3) For all $\alpha, \beta \in S$, $\alpha + \beta \notin \Sigma$.

Proof. We will construct S inductively, ensuring that $S \subseteq \Sigma^+(H)$.

First, observe that there is a partial order on $\Sigma^+(H)$ given by $\alpha \succ \beta$ if $\alpha(H) > \beta(H)$.

We begin by picking $\alpha_1 \in \Sigma^+(H)$ maximal with respect to this order (that is, with the highest possible value $\alpha_1(H)$).

Now suppose we have constructed linearly independent elements $\alpha_1, \dots, \alpha_r \in \Sigma^+(H)$, where each α_i is maximal with respect to the order among all roots outside the span of $\alpha_1, \dots, \alpha_{i-1}$. Note that $\alpha_i + \alpha_j$ is not a root for any $i, j \leq r$, since this would contradict the maximality of either α_i or α_j. If $r = \dim \mathfrak{a}$, then we are done. Otherwise, we pick $\alpha_{r+1} \in \Sigma^+(H)$ maximal with respect to the order among all roots outside the span of $\alpha_1, \dots, \alpha_r$.

At the end of this process, we have constructed the desired set S. □

A.9 An auxiliary result

This section is dedicated to a small auxiliary result used in Example 1.1.6 (4).

Proposition A.9.1 *For each $k \in \mathbb{Z}^n \setminus \{0\}$, the set*

$$\mathrm{SL}(n, \mathbb{Z})k = \{\gamma k : \gamma \in \mathrm{SL}(n, \mathbb{Z})\}$$

is infinite.

Proof. We begin by considering the special case $k = d\mathrm{e}_1 = (d, 0, \dots, 0)$ for $d \in \mathbb{Z} \setminus \{0\}$. The matrix

$$\gamma = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_{n-2} \end{pmatrix} \in \mathrm{SL}(n, \mathbb{Z})$$

which has a 2×2 block in the upper-left corner, the $(n-2) \times (n-2)$ identity matrix in the bottom-right corner, and zeroes elsewhere, satisfies

$$\gamma^m k = d\gamma^m \mathrm{e}_1 = d(\mathrm{e}_1 + m\mathrm{e}_2) = k + dm\mathrm{e}_2, \quad m \in \mathbb{Z}_{>0},$$

all distinct elements of $\mathrm{SL}(n, \mathbb{Z})k$.

Now, suppose that k is not a multiple of $\mathrm{e}_1 = (1, 0, \dots, 0)$. Let

$$J = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{SL}(n, \mathbb{Z})$$

be a $n \times n$ upper triangular Jordan matrix with ones in the diagonal. We claim that the points $J^m k \in \mathrm{SL}(n, \mathbb{Z})k$, $m \in \mathbb{Z}_{>0}$, are all distinct.

Indeed, if $J^m k = J^\ell k$ for some $m > \ell$, then $J^{m-\ell} k = k$, so k would be a 1-eigenvector of $J^{m-\ell}$. The proof will conclude once we prove that the 1-eigenspace of J^r for any $r \in \mathbb{Z}_{>0}$ is exactly $\mathbb{R}\mathrm{e}_1$, yielding a contradiction with the fact that k was assumed not to be a multiple of e_1.

To calculate the 1-eigenspace of J^r, observe that $J = \mathrm{I}_n + N$, where I_n is the $n \times n$ identity matrix and

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Observe that, in general, N^j shifts the ones to the j-th superdiagonal (so, in particular, $N^n = 0$). Then, since I_n and N commute, one can apply the binomial theorem to obtain

$$J^r = (\mathrm{I}_n + N)^r = \sum_{j=0}^{r-1} \binom{r}{j} N^j = \begin{pmatrix} \binom{r}{0} & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{n-1} \\ 0 & \binom{r}{0} & \binom{r}{1} & \cdots & \binom{r}{n-2} \\ 0 & 0 & \binom{r}{0} & \cdots & \binom{r}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{r}{0} \end{pmatrix}.$$

This forces any vector $v = (v_1, \dots, v_n)^{\mathrm{t}}$ satisfying $(J^r - \mathrm{I}_n)v = 0$ to obey the following cascade of equations:

$$\begin{aligned}
rv_2 + \binom{r}{2}v_3 + \binom{r}{3}v_4 + \cdots + \binom{r}{n-1}v_n &= 0,\\
rv_3 + \binom{r}{2}v_4 + \cdots + \binom{r}{n-2}v_n &= 0,\\
rv_4 + \binom{r}{2}v_5 + \cdots + \binom{r}{n-3}v_n &= 0,\\
&\vdots\\
rv_n &= 0,
\end{aligned}$$

which implies that $v_n = \cdots = v_2 = 0$, and so $v \in \mathbb{R}e_1$. □

Appendix B

Observations

This last chapter is a collection of three observations I made while reading Zimmer's text. They are not particularly important, and emerged only because at some points the assumptions are not stated explicitly. We explore them here, studying whether the claims hold without the implicit, unstated assumptions, for the sake of completeness.

Remark B.0.1 The author does not *explicitly* assume that the group G is Hausdorff. Some authors, like Folland [Fol16], follow the convention of saying "locally compact group" when they mean "locally compact Hausdorff group", the reason being the following: a locally compact group not being Hausdorff is not a big restriction, since then $G/\overline{\{e\}}$ would be Hausdorff (and G and $G/\overline{\{e\}}$ are measurably isomorphic: since every open set in G is $\overline{\{e\}}$-invariant, the Borel σ-algebra of G consists exactly of the inverse image of the Borel σ-algebra of $G/\overline{\{e\}}$). We're anyway mostly interested in the case of Lie groups, which are Hausdorff.

We always assume that G is Hausdorff for convenience reasons. For instance, the Riesz-Markov-Kakutani Representation Theorem requires G to be Hausdorff.

Remark B.0.2 Lemma 1.2.13, as stated in the main text, requires the assumption that $(W_k)_{k\in\mathbb{N}}$ forms a neighborhood basis at s. In Zimmer's original text [Zim84], it is not clear whether the assumption is this or that $(W_k)_{k\in\mathbb{N}}$ is a decreasing sequence of open sets with $\bigcap_{k\in\mathbb{N}} W_k = \{s\}$. We assert that the latter assumption is insufficient for the conclusion to hold.

To see why, consider $G = \mathbb{R}$ acting on $S = \mathbb{C}$ by addition. Let $N = [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, and let

$$W_k = \mathrm{B}\left(1, \frac{1}{k}\right) \cup \left(\mathrm{B}\left(\mathrm{i}, \frac{1}{k}\right) \setminus \{\mathrm{i}\}\right),$$

the union of the disk of center 1 and radius $1/k$ and the punctured disk of center i and radius $1/k$. Then, W_k is a decreasing sequence of open sets with

$\bigcap_{k\in\mathbb{N}} W_k = \{1\}$. However, $\mathrm{i} \in N + W_k$ for all $k \in \mathbb{N}$, so $\mathrm{i} \in \bigcap_{k\in\mathbb{N}}(N + W_k)$, but $\mathrm{i} \notin \mathbb{R} + 1 = \mathbb{R}$.

Remark B.0.3 In the statement of Proposition 2.1.3, the author does not *explicitly* assume second-countability of G. This creates a small issue with the fact that ergodicity is only defined for σ-finite measures, and the Haar measure of G might not be σ-finite unless G is second countable (hence the measure on G/H might not be σ-finite, think of G discrete and uncountable, and $H = \{e\}$).

Furthermore, in the proof, he uses Fubini's theorem, which requires the measures to be σ-finite. He also uses the existence of a measurable section of $G \to G/H$, which (in principle) requires G to be second countable.

Moreover, even if we extend the definition of ergodicity to include non-σ-finite measures, the claim doesn't hold. Indeed:

Let $\mathbb{R}_\mathrm{d}$ be the additive group of real numbers with the discrete topology, and $\mathbb{R}^*$ be the multiplicative group of non-zero real numbers. Let $G = \mathbb{R}_\mathrm{d} \times \mathbb{R}^*$, $H = \{0\} \times \mathbb{R}^* \simeq \mathbb{R}^* \leq G$, and $S = \mathbb{R}$ with its usual topology and the Lebesgue measure. We have that $G/H \simeq \mathbb{R}_\mathrm{d}$. Let

$$\begin{aligned} G \curvearrowright S, &\quad (x, y) \cdot s = ys, \\ \text{therefore } H \curvearrowright S, &\quad (0, y) \cdot s = ys, \\ \text{therefore } G \curvearrowright S \times G/H, &\quad (x, y) \cdot (s, t) = (ys, x + t). \end{aligned}$$

The action of H on S is ergodic, since the orbit $H \cdot 1 = \mathbb{R} \setminus \{0\}$ is conull in S. However, the action of G on $S \times G/H$ is not ergodic, since the set $\{0\} \times G/H$ is invariant, but neither null nor conull in $S \times G/H \simeq \mathbb{R} \times \mathbb{R}_\mathrm{d}$.

Remark B.0.4 In the statement of Lemma 2.2.5, the author only assumes *explicitly* that H is locally compact. This again creates a small issue with the fact that ergodicity is only defined for σ-finite measures, and the Haar measure of H might not be σ-finite unless H is second countable.

In the proof of "ergodic $\Longrightarrow$ dense", he states that $H/\overline{\Gamma}$ is metrizable. This, even though doesn't affect the proof (the proof only really uses local compactness), is not true in general unless H is locally compact and second countable (in that case, $H/\overline{\Gamma}$ would be locally compact, Hausdorff and second countable, hence Polish). Indeed, take $\Gamma = \{e\}$ and H an uncountable product of non-trivial compact Hausdorff groups, such as $H = \prod_{i\in\mathbb{R}}\{1, -1\}$. Then, $H/\overline{\Gamma} = H$ is not metrizable.

In the proof of "dense $\Longrightarrow$ ergodic", he identifies the dual of $\mathrm{L}^1(H)$ with $\mathrm{L}^\infty(H)$ and uses Fubini's theorem on $H \times A$, which requires the measure to be σ-finite (unless we observe some technicalities which here don't work, see [Fol16, §2.3]).

However, in this case, the claim is still true without second countability, and the following argument works.

Suppose $\Gamma \leq H$ is dense in H, and suppose there exists $A \subseteq H$ Borel, Γ-invariant but neither null not conull. Let m be a left Haar measure on H.

Define a Borel measure ν on H by $\nu(B) = m(A \cap B)$ for Borel $B \subseteq H$.

Claim 1. ν is Radon.

Proof of claim. For every compact $K \subseteq H$, we have $\nu(K) = m(A \cap K) \leq m(K) < \infty$.

Fix a Borel set B. To prove outer regularity for B, let $\varepsilon > 0$ and choose an open set $U \supseteq B$ with $m(U) \leq m(B) + \varepsilon$ (by outer regularity of m). Then, $\nu(U) = \nu(B) + \nu(U \setminus B) \leq \nu(B) + m(U \setminus B) \leq \nu(B) + \varepsilon$.

Fix an open set U. To prove inner regularity for U, let $\varepsilon > 0$ and choose a compact set $K \subseteq U$ with $m(K) \geq m(U) - \varepsilon$ (by inner regularity of m on open sets). Then, $\nu(U) = \nu(K) + \nu(U \setminus K) \leq \nu(K) + m(U \setminus K) \leq \nu(K) + \varepsilon$. This proves the claim.

Claim 2. ν is Γ-invariant.

Proof of claim. Let $\gamma \in \Gamma$ and let $B \subseteq H$ be Borel. Then

$$\begin{aligned} \nu(\gamma B) &= m\big(A \cap (\gamma B)\big) \\ \text{[left-invariance of } m\text{]} \quad &= m\big(\gamma^{-1}(A \cap \gamma B)\big) \\ &= m\big(\gamma^{-1}A \cap B\big) \\ \text{[}\Gamma\text{-invariance of } A\text{]} \quad &= m\big(A \cap B\big) \\ &= \nu(B), \end{aligned}$$

The claim is proven.

Claim 3. ν is H-invariant.

Proof of claim. To prove H-invariance, we will show that $\nu(g_*\varphi) = \nu(\varphi)$ for all $\varphi \in \mathrm{C_c}(H)$, viewing ν as a positive linear functional on $\mathrm{C_c}(H)$ and defining $(g_*\varphi)(x) = \varphi(g^{-1}x)$ for $g \in H$, $x \in H$.

Fix $\varphi \in \mathrm{C_c}(H)$ and define $F : H \to \mathbb{R}$ by

$$F(g) = \nu(g_*\varphi) = \int_H \varphi(g^{-1}x)\, d\nu(x),$$

which we aim to prove to be constant on H.

By Γ-invariance of ν, we have that $F(\gamma) = F(e)$ for all $\gamma \in \Gamma$. If we show that F is continuous, then F will be constant on H by density of Γ in H, hence the claim will be proven.

Fix $g_0 \in H$. Let $K = \operatorname{supp} \varphi \subseteq H$, which is compact. Let V be a compact neighborhood of $e \in H$. For $g \in g_0 V$, we have $\operatorname{supp}(g_* \varphi) = gK \subseteq g_0 V K =: C$. Note that C is compact, and in particular $\nu(C) < \infty$.

The map

$$g_0 V \times C \to \mathbb{R}, \qquad (g, x) \mapsto \varphi(g^{-1} x),$$

is continuous. By compactness of C, it follows that

$$g \mapsto \sup_{x \in C} |\varphi(g^{-1} x) - \varphi(g_0^{-1} x)|$$

is continuous as a function $g_0 V \to \mathbb{R}$.

Therefore, since $g_0 V$ is a neighborhood of g_0 in H, there exists an open neighborhood $W \subseteq g_0 V$ of g_0 (W open in H) such that

$$\sup_{x \in C} \left| \varphi(g^{-1} x) - \varphi(g_0^{-1} x) \right| \leq \frac{\varepsilon}{\max\{1, \nu(C)\}} \quad \text{for all } g \in W.$$

Hence, for $g \in W$,

$$\begin{aligned}
|F(g) - F(g_0)| &= \left| \int_C \left(\varphi(g^{-1} x) - \varphi(g_0^{-1} x) \right) d\nu(x) \right| \\
&\leq \int_C \left| \varphi(g^{-1} x) - \varphi(g_0^{-1} x) \right| \, d\nu(x) \\
&\leq \nu(C) \cdot \sup_{y \in C} \left| \varphi(g^{-1} y) - \varphi(g_0^{-1} y) \right| \\
&\leq \nu(C) \cdot \frac{\varepsilon}{\max\{1, \nu(C)\}} \\
&\leq \varepsilon.
\end{aligned}$$

This concludes the proof of the claim.

To conclude, we have that ν is Radon and H-invariant. Observe that $\nu(A) = m(A) \neq 0$ by the assumption that A is not null. Therefore, ν is nonzero, hence a left Haar measure for H. By uniqueness of Haar measures (Theorem A.3.2), we have that $\nu = c \cdot m$ for some constant $c > 0$. Since A is not conull, we have that $m(H \setminus A) > 0$. But

$$0 = m(\emptyset) = m(A \cap (H \setminus A)) = \nu(H \setminus A) = c \cdot m(H \setminus A) > 0,$$

which is a contradiction.

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